

ON CONVERGENCE OF SOLUTIONS TO EQUILIBRIA FOR QUASILINEAR PARABOLIC PROBLEMS

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ABSTRACT. We show convergence of solutions to equilibria for quasilinear parabolic evolution equations in situations where the set of equilibria is non-discrete, but forms a finite-dimensional C^1 -manifold which is normally hyperbolic. Our results do not depend on the presence of an appropriate Lyapunov functional as in the Łojasiewicz-Simon approach, but are of local nature.

Keywords: quasilinear parabolic equations, normally stable, center manifolds, nonlinear boundary conditions, free boundary problems, travelling waves.

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1. INTRODUCTION

The principle of linearized stability is a well-known and powerful tool for proving stability or instability of equilibria of nonlinear evolution equations. It is known to be true for large classes of nonlinear evolution equations, even for such which are nonlocal. The literature on this subject is large. Since here we are mainly interested in quasilinear parabolic problems, we only refer to the monograph by Lunardi [21], and to [1, 24].

In this paper we will consider the following situation: suppose that for a nonlinear evolution equation we have a C^1 -manifold of equilibria \mathcal{E} such that at a point $u_* \in \mathcal{E}$, the kernel $N(A_0)$ of the linearization A_0 is isomorphic to the tangent space of \mathcal{E} at u_* , the eigenvalue 0 of A_0 is semi-simple, and the remaining spectral part of the linearization A_0 is stable. Then solutions starting nearby u_* exist globally and converge to some point on \mathcal{E} . This result is well-known to specialists in the area of dynamical systems (where it is considered a folk theorem), but might be less familiar to people in the PDE community.

The situation described above occurs frequently in applications. We call it the *generalized principle of linearized stability*, and the equilibrium u_* is then termed *normally stable*.

A typical example for this situation to occur is the case where the equations under consideration involve symmetries, i.e. are invariant under the action of a

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Lie-group \mathcal{G} . If then u_* is an equilibrium, the manifold \mathcal{E} includes the action of \mathcal{G} on u_* and the manifold $\mathcal{G}u_*$ is a subset of \mathcal{E} .

A standard method to handle situations as described above is to refer to *center manifold theory*. In fact in that situation the center manifold of the problem in question will be unique, and it coincides with \mathcal{E} near u_* . Thus the so-called *shadowing lemma* in center manifold theory implies the result. Center manifolds are well-studied objects in the theory of nonlinear evolution equations. For the parabolic case we refer to the monographs [18, 21], and to the publications [6, 7, 10, 19, 20, 22, 28, 29].

However, the theory of center manifolds is a technically difficult matter. It usually involves higher regularity of the involved nonlinearities - in particular concerning the shadowing property. Therefore it seems desirable to have a simpler, direct approach to the generalized principle of linearized stability which avoids the technicalities of center manifold theory.

The purpose of this paper is to present such an approach. It turns out that the effort is only slightly larger than that for the proof of the standard linearized stability result - which is simple. We emphasize that our approach requires only C^1 -regularity for the nonlinearities. By several examples we will illustrate that our result is applicable to a variety of interesting problems in different areas of applied analysis. It is our belief that the approach devised in this manuscript will be fruitful for the stability analysis of equilibria for parabolic evolution equations that involve symmetries in the way described above.

Here we would also like to mention the work in [9], where the action of a Lie group has been used for the stability analysis of equilibrium solutions. However, the approach given here is considerably more general and flexible.

In Section 2 we formulate and prove our main result for abstract autonomous quasilinear parabolic problems. Theorem 2.1 implies, for instance, the main result in [15] on convergence of solutions for the Mullins-Sekerka problem. We also show by means of examples that the conditions of Theorem 2.1 are necessary in order to have convergence to a single equilibrium.

In Section 3, we consider quasilinear parabolic systems with nonlinear boundary conditions and we show that our techniques can also be applied to this situation. Sections 4 and 5 illustrate the scope of our main result, as we show convergence towards equilibria for the Mullins-Sekerka model, and stability of travelling waves for a quasilinear parabolic equation.

In Section 6 we consider the so-called *normally hyperbolic* case, where the remaining part of the spectrum of A_0 also contains an unstable part away from the imaginary axis. In this situation, one cannot expect convergence of all solutions starting near u_* , but only for those initial values which are on the stable manifold.

To cover the quasilinear case our approach makes use of maximal L_p -regularity in an essential way. As general references for this theory we refer to the recent publications [11, 12], to the survey article [24], and also to [2, 3, 4, 8, 21].

In a forthcoming paper these results are extended to the case where the boundary conditions are of relaxation type, i.e. are coupled with an evolution equation on

the boundary, as in [13]. Problems of the last kind are important e.g. for the Stefan problem with surface tension, see [14, 26], and for the two-phase Navier-Stokes problem with a free boundary.

Finally, we should like to point out that the generalized principle of linearized stability described in the current paper can also be adapted and applied to fully nonlinear parabolic equations, see [27].

2. CONVERGENCE FOR ABSTRACT QUASILINEAR PROBLEMS

Let X_0 and X_1 be two Banach spaces such that $X_1 \hookrightarrow X_0$, i.e. X_1 is continuously and densely embedded in X_0 . In this section we consider the autonomous quasilinear problem

$$\dot{u}(t) + A(u(t))u(t) = F(u(t)), \quad t > 0, \quad u(0) = u_0. \quad (2.1)$$

For $1 < p < \infty$ we introduce the real interpolation space $X_\gamma := (X_0, X_1)_{1-1/p, p}$ and we assume that there is an open set $V \subset X_\gamma$ such that

$$(A, F) \in C^1(V, \mathcal{B}(X_1, X_0) \times X_0). \quad (2.2)$$

Here $\mathcal{B}(X_1, X_0)$ denotes the space of all bounded linear operators from X_1 into X_0 . In the sequel we use the notation $|\cdot|_j$ to denote the norm in the respective spaces X_j for $j = 0, 1, \gamma$. Moreover, for any normed space X , $B_X(u, r)$ denotes the open ball in X with radius $r > 0$ around $u \in X$.

Let $\mathcal{E} \subset V \cap X_1$ denote the set of equilibrium solutions of (2.1), which means that

$$u \in \mathcal{E} \quad \text{if and only if} \quad u \in V \cap X_1, \quad A(u)u = F(u).$$

Given an element $u_* \in \mathcal{E}$, we assume that u_* is contained in an m -dimensional manifold of equilibria. This means that there is an open subset $U \subset \mathbb{R}^m$, $0 \in U$, and a C^1 -function $\Psi : U \rightarrow X_1$, such that

- $\Psi(U) \subset \mathcal{E}$ and $\Psi(0) = u_*$,
 - the rank of $\Psi'(0)$ equals m , and
 - $A(\Psi(\zeta))\Psi(\zeta) = F(\Psi(\zeta))$, $\zeta \in U$.
- (2.3)

We assume further that near u_* there are no other equilibria than those given by $\Psi(U)$, i.e. $\mathcal{E} \cap B_{X_1}(u_*, r_1) = \Psi(U)$, for some $r_1 > 0$.

We suppose that the operator $A(u_*)$ has the property of maximal L_p -regularity. Introducing the deviation $v = u - u_*$ from the equilibrium u_* , the equation for v then reads as

$$\dot{v}(t) + A_0 v(t) = G(v(t)), \quad t > 0, \quad v(0) = v_0, \quad (2.4)$$

where $v_0 = u_0 - u_*$ and

$$A_0 v = A(u_*)v + (A'(u_*)v)u_* - F'(u_*)v \quad \text{for } v \in X_1. \quad (2.5)$$

The function G can be written as $G(v) = G_1(v) + G_2(v, v)$, where

$$\begin{aligned} G_1(v) &= (F(u_* + v) - F(u_*) - F'(u_*)v) - (A(u_* + v) - A(u_*) - A'(u_*)v)u_*, \\ G_2(v, w) &= -(A(u_* + v) - A(u_*))w, \quad w \in X_1, \quad v \in V_*, \end{aligned}$$

where $V_* := V - u_*$. It follows from (2.2) that $G_1 \in C^1(V_*, X_0)$ and also that $G_2 \in C^1(V_* \times X_1, X_0)$. Moreover, we have

$$G_1(0) = G_2(0, 0) = 0, \quad G'_1(0) = G'_2(0, 0) = 0, \quad (2.6)$$

where G'_1 and G'_2 denote the Fréchet derivatives of G_1 and G_2 , respectively.

Setting $\psi(\zeta) = \Psi(\zeta) - u_*$ results in the following equilibrium equation for problem (2.4)

$$A_0\psi(\zeta) = G(\psi(\zeta)), \quad \text{for all } \zeta \in U. \quad (2.7)$$

Taking the derivative with respect to ζ and using the fact that $G'(0) = 0$ we conclude that $A_0\psi'(0) = 0$ and this implies that

$$T_{u_*}(\mathcal{E}) \subset N(A_0), \quad (2.8)$$

where $T_{u_*}(\mathcal{E})$ denotes the tangent space of \mathcal{E} at u_* .

After these preparations we can state the following result on convergence of solutions starting near u_* .

Theorem 2.1. *Let $1 < p < \infty$. Suppose $u_* \in V \cap X_1$ is an equilibrium of (2.1), and suppose that the functions (A, F) satisfy (2.2). Suppose further that $A(u_*)$ has the property of maximal L_p -regularity. Let A_0 , defined in (2.5), denote the linearization of (2.1) at u_* . Suppose that u_* is normally stable, i.e. assume that*

- (i) *near u_* the set of equilibria \mathcal{E} is a C^1 -manifold in X_1 of dimension $m \in \mathbb{N}$,*
- (ii) *the tangent space for \mathcal{E} at u_* is given by $N(A_0)$,*
- (iii) *0 is a semi-simple eigenvalue of A_0 , i.e. $N(A_0) \oplus R(A_0) = X_0$,*
- (iv) *$\sigma(A_0) \setminus \{0\} \subset \mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$.*

Then u_ is stable in X_γ , and there exists $\delta > 0$ such that the unique solution $u(t)$ of (2.1) with initial value $u_0 \in X_\gamma$ satisfying $|u_0 - u_*|_\gamma < \delta$ exists on \mathbb{R}_+ and converges at an exponential rate in X_γ to some $u_\infty \in \mathcal{E}$ as $t \rightarrow \infty$.*

Proof. (a) Note first that assumption (iii) implies that 0 is an isolated spectral point of $\sigma(A_0)$, the spectrum of A_0 . According to assumption (iv) $\sigma(A_0)$ admits a decomposition into two disjoint nontrivial parts with

$$\sigma(A_0) = \{0\} \cup \sigma_s, \quad \sigma_s \subset \mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}.$$

The spectral set $\sigma_c := \{0\}$ corresponds to the center part, and σ_s to the stable part of the analytic C_0 -semigroup $e^{-A_0 t}$, or equivalently of the Cauchy problem $\dot{w} + A_0 w = f$.

In the following, we let P^l , $l \in \{c, s\}$, denote the spectral projections according to the spectral sets $\sigma_c = \{0\}$ and σ_s , and we set $X_j^l := P^l X_j$ for $l \in \{c, s\}$ and $j \in \{0, 1, \gamma\}$. The spaces X_j^l are equipped with the norms $|\cdot|_j$ for $j = 0, 1, \gamma$. We have the topological direct decomposition

$$X_1 = X_1^c \oplus X_1^s, \quad X_0 = X_0^c \oplus X_0^s,$$

and this decomposition reduces A_0 into $A_0 = A_c \oplus A_s$, where A_l is the part of A_0 in X_0^l for $l \in \{c, s\}$. Since $\sigma_c = \{0\}$ is compact it follows that $X_0^c \subset X_1$.

Therefore, X_0^c and X_1^c coincide as vector spaces. In the following, we will just write $X^c = (X^c, |\cdot|_j)$ for either of the spaces X_0^c and X_1^c . We note that $N(A_0)$, the kernel of A_0 , is contained in X^c . The operator A_s inherits the property of L_p -maximal regularity from A_0 . Since $\sigma(A_s) = \sigma_s \subset \mathbb{C}_+$ we obtain that the Cauchy problem

$$\dot{w} + A_s w = f, \quad w(0) = 0, \quad (2.9)$$

also enjoys the property of maximal regularity, even on the interval $J = (0, \infty)$. In fact the following estimates are true. For any $a \in (0, \infty]$ let

$$\mathbb{E}_0(a) = L_p((0, a); X_0), \quad \mathbb{E}_1(a) = H_p^1((0, a); X_0) \cap L_p((0, a); X_1). \quad (2.10)$$

The natural norms in $\mathbb{E}_j(a)$ will be denoted by $\|\cdot\|_{\mathbb{E}_j(a)}$ for $j = 0, 1$. Then the Cauchy problem (2.9) has for each $f \in L_p((0, a); X_0^s)$ a unique solution

$$w \in H_p^1((0, a); X_0^s) \cap L_p((0, a); X_1^s),$$

and there exists a constant M_0 such that $\|w\|_{\mathbb{E}_1(a)} \leq M_0 \|f\|_{\mathbb{E}_0(a)}$ for every $a > 0$, and every function $f \in L_p((0, a); X_0^s)$. In fact, since $\sigma(A_s - \omega)$ is still contained in \mathbb{C}_+ for ω small enough, we see that the operator $A_s - \omega$ enjoys the same properties as A_s . Therefore, every solution of the Cauchy problem (2.9) satisfies the estimate

$$\|e^{\sigma t} w\|_{\mathbb{E}_1(a)} \leq M_0 \|e^{\sigma t} f\|_{\mathbb{E}_0(a)}, \quad \sigma \in [0, \omega], \quad a > 0, \quad (2.11)$$

for $f \in L_p((0, a); X_0^s)$, where $M_0 = M_0(\omega)$ for $\omega > 0$ fixed; cf. [24, Sec. 6]. Furthermore, there exists a constant $M_1 > 0$ such that

$$\|e^{\omega t} e^{-A_s t} P^s u\|_{\mathbb{E}_1(a)} + \sup_{t \in [0, a]} |e^{\omega t} e^{-A_s t} P^s u|_\gamma \leq M_1 |P^s u|_\gamma \quad (2.12)$$

for every $u \in X_\gamma$ and $a \in (0, \infty]$. For future use we note that

$$\sup_{t \in [0, a]} |w(t)|_\gamma \leq c_0 \|w\|_{\mathbb{E}_1(a)} \quad \text{for all } w \in \mathbb{E}_1(a) \text{ with } w(0) = 0 \quad (2.13)$$

with a constant c_0 that is independent of $a \in (0, \infty]$, see for instance the proof of [25, Proposition 6.2]. We remind that $N(A_0)$ is contained in X^c .

(b) It follows from the considerations above and assumptions (i)-(iii) that in fact

$$N(A_0) = X^c \quad \text{and} \quad \dim(X^c) = m.$$

As X^c has finite dimension, the norms $|\cdot|_j$ for $j = 0, 1, \gamma$ are equivalent, and we equip X^c with one of these equivalent norms, say with $|\cdot|_0$. Let us now consider the mapping

$$g : U \subset \mathbb{R}^m \rightarrow X^c, \quad g(\zeta) := P^c \psi(\zeta), \quad \zeta \in U.$$

It follows from our assumptions that $g'(0) = P^c \psi'(0) : \mathbb{R}^m \rightarrow X^c$ is an isomorphism (between the finite dimensional spaces \mathbb{R}^m and X^c). By the inverse function theorem, g is a C^1 -diffeomorphism of a neighborhood of 0 in \mathbb{R}^m into a neighborhood, say $B_{X^c}(0, \rho_0)$, of 0 in X^c . Let $g^{-1} : B_{X^c}(0, \rho_0) \rightarrow U$ be the inverse mapping. Then $g^{-1} : B_{X^c}(0, \rho_0) \rightarrow U$ is C^1 and $g^{-1}(0) = 0$. Next we set $\Phi(x) := \psi(g^{-1}(x))$ for $x \in B_{X^c}(0, \rho_0)$ and we note that

$$\Phi \in C^1(B_{X^c}(0, \rho_0), X_1^s), \quad \Phi(0) = 0, \quad \{\Phi(x) + u_* : x \in B_{X^c}(0, \rho_0)\} = \mathcal{E} \cap W,$$

where W is an appropriate neighborhood of u_* in X_1 . One readily verifies that

$$P^c\Phi(x) = ((P^c \circ \psi) \circ g^{-1})(x) = (g \circ g^{-1})(x) = x, \quad x \in B_{X^c}(0, \rho_0),$$

and this yields $\Phi(x) = P^c\Phi(x) + P^s\Phi(x) = x + P^s\Phi(x)$ for $x \in B_{X^c}(0, \rho_0)$. Setting $\phi(x) := P^s\Phi(x)$ we conclude that

$$\phi \in C^1(B_{X^c}(0, \rho_0), X_1^s), \quad \phi(0) = \phi'(0) = 0, \quad (2.14)$$

and that

$$\{x + \phi(x) + u_* : x \in B_{X^c}(0, \rho_0)\} = \mathcal{E} \cap W,$$

where W is a neighborhood of u_* in X_1 . This shows that the manifold \mathcal{E} can be represented as the (translated) graph of the function ϕ in a neighborhood of u_* . Moreover, the tangent space of \mathcal{E} at u_* coincides with $N(A_0) = X^c$. By applying the projections P^l , $l \in \{c, s\}$, to equation (2.7) and using that $x + \phi(x) = \psi(g^{-1}(x))$ for $x \in B_{X^c}(0, \rho_0)$, and that $A_c \equiv 0$, we obtain the following equivalent system of equations for the equilibria of (2.4)

$$P^cG(x + \phi(x)) = 0, \quad P^sG(x + \phi(x)) = A_s\phi(x), \quad x \in B_{X^c}(0, \rho_0). \quad (2.15)$$

Finally, let us also agree that ρ_0 has already been chosen small enough so that

$$|\phi'(x)|_{\mathcal{B}(X^c, X_1^s)} \leq 1, \quad |\phi(x)|_1 \leq |x|, \quad x \in B_{X^c}(0, \rho_0). \quad (2.16)$$

This can always be achieved, thanks to (2.14).

(c) Introducing the new variables

$$\begin{aligned} x &= P^c v = P^c(u - u_*), \\ y &= P^s v - \phi(P^c v) = P^s(u - u_*) - \phi(P^c(u - u_*)) \end{aligned}$$

we then obtain the following system of evolution equations in $X^c \times X_0^s$

$$\begin{cases} \dot{x} = T(x, y), & x(0) = x_0, \\ \dot{y} + A_s y = R(x, y), & y(0) = y_0, \end{cases} \quad (2.17)$$

with $x_0 = P^c v_0$ and $y_0 = P^s v_0 - \phi(P^c v_0)$, where the functions T and R are given by

$$\begin{aligned} T(x, y) &= P^cG(x + \phi(x) + y), \\ R(x, y) &= P^sG(x + \phi(x) + y) - A_s\phi(x) - \phi'(x)T(x, y). \end{aligned}$$

Using the equilibrium equations (2.15), the expressions for R and T can be rewritten as

$$\begin{aligned} T(x, y) &= P^c(G(x + \phi(x) + y) - G(x + \phi(x))), \\ R(x, y) &= P^s(G(x + \phi(x) + y) - G(x + \phi(x))) - \phi'(x)T(x, y). \end{aligned} \quad (2.18)$$

Although the term $P^cG(x + \phi(x))$ in T is zero, see (2.15), we include it here for reasons of symmetry, and for justifying the estimates for T below. Equation (2.18) immediately yields

$$T(x, 0) = R(x, 0) = 0 \quad \text{for all } x \in B_{X^c}(0, \rho_0),$$

showing that the equilibrium set \mathcal{E} of (2.1) near u_* has been reduced to the set $B_{X^c}(0, \rho_0) \times \{0\} \subset X^c \times X_1^s$.

Observe also that there is a unique correspondence between the solutions of (2.1) close to u_* in X_γ and those of (2.17) close to 0. We call system (2.17) the *normal form* of (2.1) near its normally stable equilibrium u_* .

(d) From the representation of G and (2.6) we obtain the following estimates for G_1 and G_2 : for given $\eta > 0$ we may choose $r = r(\eta) > 0$ small enough such that

$$|G_1(v_1) - G_1(v_2)|_0 \leq \eta |v_1 - v_2|_\gamma, \quad v_1, v_2 \in B_{X_\gamma}(0, r).$$

Moreover, there is a constant $L > 0$ such that

$$\begin{aligned} |G_2(v_1, w) - G_2(v_2, w)|_0 &\leq L |w|_1 |v_1 - v_2|_\gamma, \quad w \in X_1, \quad v_1, v_2 \in B_{X_\gamma}(0, r), \\ |G_2(v, w_1) - G_2(v, w_2)|_0 &\leq L r |w_1 - w_2|_1, \quad w_1, w_2 \in X_1, \quad v \in B_{X_\gamma}(0, r). \end{aligned}$$

We remark that L does not depend on $r \in (0, r_0]$ with r_0 appropriately chosen. Combining these estimates we have

$$\begin{aligned} |G(v_1) - G(v_2)|_0 &\leq (\eta + L |v_2|_1) |v_1 - v_2|_\gamma + L r |v_1 - v_2|_1 \\ &\leq C_0 (\eta + r + |v_2|_1) |v_1 - v_2|_1 \end{aligned} \quad (2.19)$$

for all $v_1, v_2 \in B_{X_\gamma}(0, r) \cap X_1$, where C_0 is independent of $r \in (0, r_0]$.

In the following, we will always assume that $r \in (0, r_0]$ and $r_0 \leq 3\rho_0$. Taking $v_1 = x + \phi(x) + y$ and $v_2 = x + \phi(x)$ in (2.19) we infer from (2.16) and (2.18) that

$$|T(x, y)|, |R(x, y)|_0 \leq C_1 (\eta + r + |x + \phi(x)|_1) |y|_1 \leq \beta |y|_1, \quad (2.20)$$

for all $x \in \bar{B}_{X^c}(0, \rho)$, $y \in \bar{B}_{X_\gamma^s}(0, \rho) \cap X_1$ and all $\rho \in (0, r/3)$, where $\beta = C_2(\eta + r)$, and where C_1 and C_2 are uniform constants. Suppose that η and, accordingly, r were already chosen small enough so that

$$M_0 \beta = M_0 C_2 (\eta + r) \leq 1/2. \quad (2.21)$$

(e) By [24, Theorem 3.1], problem (2.4) admits for each $v_0 \in B_{X_\gamma}(0, r)$ a unique local strong solution

$$v \in \mathbb{E}_1(a) \cap C([0, a]; X_\gamma) \quad (2.22)$$

for some number $a > 0$. This solution can be extended to a maximal interval of existence $[0, t_*)$. If t_* is finite, then either $v(t)$ leaves the ball $B_{X_\gamma}(0, r)$ at time t_* , or the limit $\lim_{t \rightarrow t_*} v(t)$ does not exist in X_γ . We show that this cannot happen for initial values $v_0 \in B_{X_\gamma}(0, \delta)$, with $\delta \leq r$ to be chosen later.

Suppose that $x_0 \in B_{X^c}(0, N\delta)$ and $y_0 \in B_{X_\gamma^s}(0, N\delta)$ are given, where the number δ will be determined later and $N := \|P^c\|_{\mathcal{B}(X_0)} + \|P^s\|_{\mathcal{B}(X_\gamma)}$. Let t_* denote the existence time for the solution $(x(t), y(t))$ of system (2.17) with initial values (x_0, y_0) , or equivalently, for the solution $v(t)$ of (2.4) with initial value $v_0 = x_0 + \phi(x_0) + y_0$. Let ρ be fixed so that the estimates in (2.20) hold. Set

$$t_1 := t_1(x_0, y_0) := \sup\{t \in (0, t_*) : |x(\tau)|, |y(\tau)|_\gamma \leq \rho, \tau \in [0, t]\}$$

and suppose that $t_1 < t_*$. Due to (2.11)–(2.12) and (2.20) we obtain

$$\begin{aligned} \|e^{\omega t} y\|_{\mathbb{E}_1(t_1)} &\leq M_1 |y_0|_\gamma + M_0 \|e^{\omega t} R(x, y)\|_{\mathbb{E}_0(t_1)} \\ &\leq M_1 |y_0|_\gamma + M_0 \beta \|e^{\omega t} y\|_{\mathbb{E}_1(t_1)}. \end{aligned}$$

This yields with (2.21)

$$\|e^{\sigma t} y\|_{\mathbb{E}_1(t_1)} \leq 2M_1 |y_0|_\gamma, \quad \sigma \in [0, \omega]. \quad (2.23)$$

Using this estimate as well as (2.12)–(2.13) we further have for $t \in [0, t_1]$

$$\begin{aligned} |e^{\sigma t} y(t)|_\gamma &\leq |e^{\sigma t} y(t) - e^{\sigma t} e^{-A_s t} y_0|_\gamma + |e^{\sigma t} e^{-A_s t} y_0|_\gamma \\ &\leq c_0 \|e^{\sigma t} y(t) - e^{\sigma t} e^{-A_s t} y_0\|_{\mathbb{E}_1(t_1)} + M_1 |y_0|_\gamma \\ &\leq (3c_0 M_1 + M_1) |y_0|_\gamma, \end{aligned}$$

which yields with $M_2 = (3c_0 + 1)M_1$,

$$|y(t)|_\gamma \leq M_2 e^{-\sigma t} |y_0|_\gamma, \quad t \in [0, t_1], \quad \sigma \in [0, \omega]. \quad (2.24)$$

We deduce from the equation for x , the estimate for T in (2.20), and Hölder's inequality that

$$\begin{aligned} |x(t)| &\leq |x_0| + \int_0^t |T(x(s), y(s))| ds \\ &\leq |x_0| + \beta \int_0^t |y(s)|_1 ds \\ &\leq |x_0| + \beta \left(\int_0^\infty e^{-\omega s p'} ds \right)^{1/p'} \|e^{\omega t} y\|_{\mathbb{E}_1(t_1)} \\ &= |x_0| + \beta c_1 \|e^{\omega t} y\|_{\mathbb{E}_1(t_1)} \leq |x_0| + M_3 |y_0|_\gamma, \quad t \in [0, t_1], \end{aligned}$$

where $M_3 = M_1 c_1 / M_0$ and $c_1 = (1/[\omega p'])^{1/p'}$. Summarizing, we have shown that $|x(t)| + |y(t)|_\gamma \leq |x_0| + (M_2 + M_3) |y_0|_\gamma$ for all $t \in [0, t_1]$. By continuity and the assumption $t_1 < t_*$ this inequality also holds for $t = t_1$. Hence

$$|x(t_1)| + |y(t_1)|_\gamma \leq |x_0| + (M_2 + M_3) |y_0|_\gamma \leq (1 + M_2 + M_3) N \delta < \rho/2,$$

provided $\delta \leq \rho/[2N(1 + M_2 + M_3)]$. This contradicts the definition of t_1 and we conclude that $t_1 = t_*$.

In the following, we assume that $\delta \leq \rho/[2N(1 + M_2 + M_3)]$. Then the estimates derived above and (2.16) yield the uniform bounds

$$\|v\|_{\mathbb{E}_1(a)} + \sup_{t \in [0, a]} |v(t)|_\gamma \leq M, \quad (2.25)$$

for every initial value $v_0 \in B_{X_\gamma}(0, \delta)$ and every $a < t_*$. It follows from Corollary 3.2 in [24] that the solution $v(t)$ of (2.4) exists on \mathbb{R}_+ .

(f) By repeating the above estimates on the interval $(0, \infty)$ we obtain the estimates

$$|x(t)| \leq |x_0| + M_3 |y_0|_\gamma, \quad |y(t)|_\gamma \leq M_2 e^{-\omega t} |y_0|_\gamma, \quad t \in [0, \infty), \quad (2.26)$$

for all $x_0 \in B_{X^c}(0, N\delta)$ and $y_0 \in B_{X_\gamma^s}(0, N\delta)$. Moreover,

$$\lim_{t \rightarrow \infty} x(t) = x_0 + \int_0^\infty T(x(s), y(s)) ds =: x_\infty$$

exists since the integral is absolutely convergent. Next observe that we in fact obtain exponential convergence of $x(t)$ towards x_∞ , as

$$\begin{aligned} |x(t) - x_\infty| &= \left| \int_t^\infty T(x(s), y(s)) ds \right| \\ &\leq \beta \int_t^\infty |y(s)|_1 ds \\ &\leq \beta \left(\int_t^\infty e^{-\omega s p'} ds \right)^{1/p'} \|e^{\omega s} y\|_{\mathbb{E}_1(\infty)} \\ &\leq M_4 e^{-\omega t} |y_0|_\gamma, \quad t \geq 0. \end{aligned}$$

This yields existence of

$$v_\infty := \lim_{t \rightarrow \infty} v(t) = \lim_{t \rightarrow \infty} x(t) + \phi(x(t)) + y(t) = x_\infty + \phi(x_\infty).$$

Clearly, v_∞ is an equilibrium for equation (2.4), and $v_\infty + u_* \in \mathcal{E}$ is an equilibrium for (2.2). Due to (2.16), (2.26) and the exponential estimate for $|x(t) - x_\infty|$ we get

$$\begin{aligned} |v(t) - v_\infty|_\gamma &= |x(t) + \phi(x(t)) + y(t) - v_\infty|_\gamma \\ &\leq |x(t) - x_\infty|_\gamma + |\phi(x(t)) - \phi(x_\infty)|_\gamma + |y(t)|_\gamma \\ &\leq (CM_4 + M_2) e^{-\omega t} |y_0|_\gamma \\ &\leq M e^{-\omega t} |P^s v_0 - \phi(P^c v_0)|_\gamma, \end{aligned} \tag{2.27}$$

thereby completing the proof of the second part of Theorem 2.1. Concerning stability, note that given $r > 0$ small enough we may choose $0 < \delta \leq r$ such that the solution starting in $B_{X_\gamma}(u_*, \delta)$ exists on \mathbb{R}_+ and stays within $B_{X_\gamma}(u_*, r)$. \square

Remarks 2.2. (a) Theorem 2.1 shows, given that situation, that near u_* the set of equilibria constitutes the (unique) center manifold for (2.1).

(b) It is worthwhile to point out a slightly different way to obtain the function ϕ used in the proof of Theorem 2.1. Applying the projections P^s and P^c to the equilibrium equation (2.7) yields the following equivalent system of equations near $v = 0$

$$A_s z = P^s G(x + z), \quad A_c x = P^c G(x + z), \tag{2.28}$$

with $z = P^s \psi(\zeta)$ and $x = P^c \psi(\zeta)$. Since $G(0) = G'(0) = 0$ and A_s is invertible, by the implicit function theorem we may solve the first equation for z in terms of x , i.e. there is a C^1 -function $\phi : B_{X^c}(0, \rho_0) \rightarrow X_1^s$ such that

$$\phi(0) = 0 \quad \text{and} \quad A_s \phi(x) = P^s G(x + \phi(x)), \quad x \in B_{X^c}(0, \rho_0).$$

As $x + \phi(x)$ is the unique solution of the first equation in (2.28) we additionally have $A_c x = P^c G(x + \phi(x))$, as well as $P^s \psi(\zeta) = \phi(P^c \psi(\zeta))$ for all $\zeta \in U$. Since

$G'(0) = 0$ we obtain $A_s \phi'(0) = P^s G'(0) = 0$ and this implies $\phi'(0) = 0$. This shows that $\mathcal{E} \subset \mathcal{M}$ with $\mathcal{M} = \{x + \phi(x) + u_* : x \in B_{X^c}(0, \rho_0)\}$ in a neighborhood of u_* in X_1 .

\mathcal{M} is a C^1 -manifold of dimension $\ell := \dim(X^c)$ with tangent space $T_{u_*}(\mathcal{M}) = X^c$ and \mathcal{E} is a submanifold in \mathcal{M} . In general, \mathcal{E} has lower dimension than \mathcal{M} . Our assumptions in Theorem 2.1 do in fact exactly amount to asserting that \mathcal{E} and \mathcal{M} are of equal dimension. Since $\mathcal{E} \subset \mathcal{M}$ we can then conclude that they coincide in a neighborhood of u_* .

(c) An inspection of the argument given above shows that in fact all equilibria of equation (2.1) that are close to the equilibrium u_* are contained in a manifold $\mathcal{M} = \{x + \phi(x) + u_* : x \in B_{X^c}(0, \rho_0)\}$ such that $\phi(0) = \phi'(0) = 0$, with no additional assumptions on the structure of the equilibria. To see this, let us once more consider the equation

$$A_s z = P^s G(x + z), \quad x \in X^c, \quad z \in X_1^s. \quad (2.29)$$

Clearly, $x = z = 0$ is a solution. Exactly as in the remark above, we can solve (2.29) by the implicit function theorem for z in terms of x , obtaining a C^1 -function $\phi : B_{X^c}(0, \rho_0) \rightarrow X_1^s$ with $\phi(0) = \phi'(0) = 0$. If $v \in X_1$ is an equilibrium for the evolution equation (2.4) close to 0, then the pair $x = P^c v$, $z = P^s v$ necessarily satisfies equation (2.29), and therefore lies on the graph of ϕ .

(d) We illustrate by means of examples that convergence to equilibria fails if one of the conditions (i)-(iii) in Theorem 2.1 does not hold.

Example 1. Consider in $G := \mathbb{R}^2 \setminus \{0\}$ the ODE system

$$\begin{aligned} \dot{x} &= (x + y)(1 - \sqrt{x^2 + y^2}), \\ \dot{y} &= (y - x)(1 - \sqrt{x^2 + y^2}). \end{aligned} \quad (2.30)$$

In polar coordinates (2.30) reads as

$$\begin{aligned} \dot{r} &= -r(r - 1), \\ \dot{\theta} &= r - 1, \end{aligned}$$

thus the set of equilibria \mathcal{E} of (2.30) in G is the unit circle, and for any initial value $(x_0, y_0) \in G$ we have $r(t) \rightarrow 1$ as $t \rightarrow \infty$. Since the phase portrait is rotationally invariant we may restrict the stability analysis for \mathcal{E} to one equilibrium, say $u_* = (0, 1)$. Denoting the right-hand side of (2.30) by $F(x, y)$, we have $F \in C^1(G)$, and

$$A_0 = -F'(u_*) = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

The eigenvalues of A_0 are 0 and 1 with eigenvectors $(1, 0)$ and $(1, 1)$, respectively. Thus 0 is semi-simple, and $N(A_0)$ coincides with the tangent space $T_{u_*}(\mathcal{E})$. Consequently, u_* is normally stable, and hence we can apply Theorem 2.1 to conclude that each trajectory converges to some point on the unit circle as $t \rightarrow \infty$. It is readily seen that the trajectories satisfy the relation $\theta(r) = c_0 - \ln r$ for some appropriate constant c_0 , and this confirms that $\theta(r)$ converges as $r \rightarrow 1$.

Example 2. In this example, we consider in $G := \mathbb{R}^2 \setminus \{0\}$ the ODE system

$$\begin{aligned}\dot{x} &= -x(\sqrt{x^2 + y^2} - 1)^3 - y(\sqrt{x^2 + y^2} - 1)^m, \\ \dot{y} &= -y(\sqrt{x^2 + y^2} - 1)^3 + x(\sqrt{x^2 + y^2} - 1)^m,\end{aligned}\tag{2.31}$$

with $m = 1$. In polar coordinates (2.31) reads as

$$\begin{aligned}\dot{r} &= -r(r - 1)^3, \\ \dot{\theta} &= (r - 1)^m.\end{aligned}$$

Again, the set of equilibria \mathcal{E} of (2.31) in G is the unit circle. As above, we may restrict the stability analysis for \mathcal{E} to one equilibrium, say $u_* = (0, 1)$. Denoting the right side of (2.31) by $F(x, y)$ we obtain (in case $m = 1$)

$$A_0 = -F'(u_*) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Clearly, $\{0\}$ is an eigenvalue of A_0 with algebraic multiplicity 2, and $N(A_0) = \text{span}\{(1, 0)\}$. Therefore, the eigenvalue $\{0\}$ has geometric multiplicity 1 and algebraic multiplicity 2. So we have the following situation:

- (i) $\mathcal{E} = \{(x, y) : x^2 + y^2 = 1\}$ is a smooth manifold of dimension 1 in \mathbb{R}^2 ,
- (ii) the tangent space of \mathcal{E} at u_* is given by $N(A_0)$,
- (iii) $\{0\}$ is *not* semi-simple,

and hence condition (iii) of Theorem 2.1 is not satisfied. We will show that the trajectories of system (2.31) still converge towards the unit circle, but will spiral around the circle at increasing speed as $r \rightarrow 1$. This can be seen as follows. First we observe that $V(x, y) := (r - 1)^2$ with $r = \sqrt{x^2 + y^2}$ is a Lyapunov function for system (2.31), since for every solution (x, y) of (2.31) we have

$$\frac{d}{dt}V(x, y) = 2\dot{r}(r - 1) = -2r(r - 1)^4 \leq 0.$$

So $r(t) \rightarrow 1$ as $t \rightarrow \infty$ for every solution. On the other hand one verifies that the trajectories satisfy the relation $\theta(r) = c_0 + \ln(|r - 1|/r) + 1/(r - 1)$. This shows that all trajectories spiral around \mathcal{E} with increasing speed, in clockwise direction as $r \nearrow 1$, and in counter-clockwise direction as $r \searrow 1$.

Example 3: Here we consider system (2.31) with $m = 2$. This example is similar to the one in [5, p. 4]. In this case we have

$$A_0 = -F'(u_*) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Clearly, $\{0\}$ is now an eigenvalue with geometric multiplicity 2, and $N(A_0) = \mathbb{R}^2$. So condition (ii) of Theorem 2.1 is not satisfied. The function V from the previous example is again a Lyapunov function, and this yields $r(t) \rightarrow 1$ as $t \rightarrow \infty$. The trajectories satisfy $\theta(r) = c_0 - \ln(|r - 1|/r)$, showing that they spiral counter-clockwise with increasing speed around the unit circle as $r \rightarrow 1$.

3. QUASILINEAR PARABOLIC PROBLEMS WITH NONLINEAR BOUNDARY CONDITIONS

The analysis in the previous section applies in particular to quasilinear parabolic systems of partial differential equations with *linear autonomous* boundary conditions. In this section we show how this can be extended to the case where also the boundary conditions are nonlinear. For this purpose, let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with boundary $\partial\Omega \in C^{2m}$. The outer normal at a point $x \in \partial\Omega$ will be denoted by $\nu(x)$. Consider the problem

$$\begin{cases} \partial_t u(t) + A(u(t))u(t) = F(u(t)) & \text{in } \Omega, \\ B_j(u(t)) = 0 & \text{on } \partial\Omega, \quad j \in \{1, \dots, m\}, \\ u(0) = u_0 & \text{in } \Omega. \end{cases} \quad (3.1)$$

Here we employ the maps

$$\begin{aligned} [A(u)v](x) &= \sum_{|\alpha|=2m} a_\alpha(x, u(x), \nabla u(x), \dots, \nabla^{2m-1}u(x)) D^\alpha v(x), \quad x \in \Omega, \\ [F(u)](x) &= f(x, u(x), \nabla u(x), \dots, \nabla^{2m-1}u(x)), \quad x \in \Omega, \\ [B_j(u)](x) &= b_j(x, u(x), \nabla u(x), \dots, \nabla^{m_j}u(x)), \quad x \in \partial\Omega, \end{aligned} \quad (3.2)$$

for functions $u \in BC^{2m-1}(\overline{\Omega}; \mathbb{C}^N)$, and $v \in W_p^{2m}(\Omega; \mathbb{C}^N)$. The numbers m_j are integers strictly smaller than $2m$, and with $E = \mathbb{C}^N$, the coefficients are subject to the following regularity assumptions

$$\begin{aligned} \text{(R)} \quad a_\alpha &\in C^1(E \times E^N \times \dots \times E^{N^{2m-1}}; BC(\overline{\Omega}; \mathcal{B}(E))) \text{ for } |\alpha| = 2m, \\ f &\in C^1(E \times E^N \times \dots \times E^{N^{2m-1}}; BC(\overline{\Omega}; E)), \\ b_j &\in C^{2m+1-m_j}(\partial\Omega \times E \times E^N \times \dots \times E^{N^{m_j}}; E) \text{ for } j \in \{1, \dots, m\}. \end{aligned}$$

We set $B = (B_1, \dots, B_m)$. We point out that, for a fixed $u_0 \in BC^{2m-1}(\overline{\Omega}; \mathbb{C}^N)$, $A(u_0)$ is a linear differential operator of order $2m$ with bounded coefficients; whereas F contains all terms involving derivatives of order $|\alpha| < 2m$.

We will employ the L_p -setting for this problem as in [19], hence we fix $p > n+2m$ and the basic spaces

$$X_0 = L_p(\Omega; E), \quad X_1 = W_p^{2m}(\Omega; E), \quad X_\gamma = (X_0, X_1)_{1-1/p} = W_p^{2m(1-1/p)}(\Omega; E).$$

As in Section 2 we denote the norm in X_j by $|\cdot|_j$ and open balls in X_j by $B_{X_j}(u, r)$, $j = 0, 1, \gamma$. Note that by the Sobolev embedding theorem we have $X_\gamma \hookrightarrow BC^{2m-1}(\overline{\Omega}; E)$, which allows us to plug in functions $u \in X_\gamma$ into the coefficients of A , into f and into the functions b_j pointwise, without any growth restrictions on these nonlinearities.

Assume we have a C^1 -manifold of equilibria $\Psi : U \rightarrow X_1$ where $U \subset \mathbb{R}^k$ is an open neighborhood of 0,

$$\begin{aligned} A(\Psi(\zeta))\Psi(\zeta) &= F(\Psi(\zeta)) & \text{in } \Omega, \quad \zeta \in U, \\ B(\Psi(\zeta)) &= 0, & \text{on } \partial\Omega, \quad \zeta \in U, \end{aligned} \quad (3.3)$$

and set $u_* = \Psi(0)$. Assume that the rank of $\Psi'(0)$ is k and that there are no other equilibria near u_* in X_1 , i.e. $\mathcal{E} \cap B_{X_1}(u_*, r_1) = \Psi(U)$, for some $r_1 > 0$, where as in Section 2, \mathcal{E} denotes the set of equilibria of (3.1).

The linearization of (3.1) at u_* is given by the operator A_0 defined as follows

$$\begin{aligned} A_* v &= A(u_*)v + (A'(u_*)v)u_* - F'(u_*)v && \text{in } \Omega, \\ B_* v &= B'(u_*)v && \text{on } \partial\Omega, \\ \text{with } v &\in D(A_*) = D(B_*) = W_p^{2m}(\Omega; E), \\ A_0 &= A_*|_{N(B_*)}. \end{aligned} \tag{3.4}$$

Next we consider the property of maximal L_p -regularity for the pair (A_*, B_*) , and in particular for the operator A_0 . For this we only need to consider the principal parts of the corresponding differential operator and of the boundary operators, i.e.

$$\begin{aligned} A_{\#}(x, D) &= \sum_{|\alpha|=2m} a_{\alpha}(x, u_*(x), \dots, \nabla^{2m-1}u_*(x))D^{\alpha}, \\ B_{j\#}(x, D) &= \sum_{|\beta|=m_j} i^{m_j} [\partial b_j / \partial (\partial_x^{\beta} u)](x, \dots, \nabla^{2m-1}u_*(x))D^{\beta}, \end{aligned}$$

for $j = 1, \dots, m$. Note that we use the notation $D = -i\nabla$, hence $\nabla^{\beta} = i^{|\beta|}D^{\beta}$. It is shown in [12] that normal ellipticity of $A_{\#}$ and the Lopatinskii-Shapiro condition for $(A_{\#}, B_{\#})$ are necessary, and in [11] that they are also sufficient for L_p -maximal regularity of A_0 . These conditions read as follows.

(E) For all $x \in \bar{\Omega}$, $\xi \in \mathbb{R}^n$, $|\xi| = 1$, $\sigma(\mathcal{A}_{\#}(x, \xi)) \subset \mathbb{C}_+$,
i.e. $\mathcal{A}(x, D)$ is **normally elliptic**.

(LS) For all $x \in \partial\Omega$, $\xi \in \mathbb{R}^n$, with $\xi \cdot \nu(x) = 0$, $\lambda \in \overline{\mathbb{C}_+}$, $\lambda \neq 0$, and $h \in E^m$,
the system of ordinary differential equations on the half-line

$$\begin{aligned} \lambda v(y) + \mathcal{A}_{\#}(x, \xi + i\nu(x)\partial_y)v(y) &= 0, \quad y > 0, \\ \mathcal{B}_{j\#}(x, \xi + i\nu(x)\partial_y)v(0) &= h_j, \quad j = 1, \dots, m, \end{aligned}$$

admits a unique solution $v \in C_0(\mathbb{R}_+; E)$.

(Lopatinskii-Shapiro condition.)

Now assume that $u_* \in X_1$ is an equilibrium of (3.1), and let conditions (R), (E), and (LS) be satisfied. It was shown in [19] that (3.1) then admits a local strong solution in the L_p -sense for each initial value $u_0 \in X_{\gamma}$, provided the compatibility condition $B(u_0) = 0$ holds and $|u_0 - u_*|_{\gamma}$ is sufficiently small. The solution map $[u_0 \mapsto u(t, u_0)]$ defines a local semi-flow in X_{γ} near u_* on the nonlinear phase-manifold

$$\mathcal{M} = \{u \in X_{\gamma} : B(u) = 0 \text{ on } \partial\Omega\}.$$

In case the equilibrium u_* is *hyperbolic*, i.e. $\sigma(A_0) \cap i\mathbb{R} = \emptyset$, it was moreover shown in [19] that it is isolated and that it has the so-called *saddle point property*, which means that the local semi-flow in \mathcal{M} admits a unique stable and unstable manifold

near u_* . We refer to [19] for details as well as to [24] in the case of linear boundary conditions.

Returning to our situation, differentiating (3.3) w.r.t. ζ we obtain for $\zeta = 0$

$$\begin{aligned} A(u_*)\Psi'(0) + [A'(u_*)\Psi'(0)]u_* - F'(u_*)\Psi'(0) &= 0 \quad \text{in } \Omega, \\ B'_j(u_*)\Psi'(0) &= 0 \quad \text{on } \partial\Omega, \quad j = 1, \dots, m. \end{aligned}$$

This shows that the image of $\Psi'(0)$ is contained in the kernel $N(A_0)$ of A_0 , and also that $T_{u_*}(\mathcal{E})$, the tangential space of \mathcal{E} at u_* , is contained in $N(A_0)$. As in Section 2 we assume now that $R(\Psi'(0)) = N(A_0)$, that the eigenvalue 0 of A_0 is semi-simple, and that the remaining spectrum of A_0 is contained in the open right half-plane \mathbb{C}_+ . Note that by boundedness of Ω and compact embedding, the spectrum of A_0 consists only of isolated eigenvalues of finite algebraic multiplicity, anyway. We can now state the main result of this section.

Theorem 3.1. *Let $2m + n < p < \infty$, let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with boundary of class C^{2m} , and let the spaces X_j , $j = 0, 1, \gamma$, be defined as above. Suppose $u_* \in X_1$ is an equilibrium of (3.1), and assume that conditions (R), (E), and (LS) are satisfied. Let A_0 defined in (3.4) denote the linearization of (3.1) at u_* , and suppose that u_* is normally stable, i.e. assume that*

- (i) *near u_* the set of equilibria \mathcal{E} is a C^1 -manifold in X_1 of dimension $k \in \mathbb{N}$,*
- (ii) *the tangent space for \mathcal{E} at u_* is given by $N(A_0)$,*
- (iii) *0 is a semi-simple eigenvalue of A_0 , i.e. $R(A_0) \oplus N(A_0) = X_0$,*
- (iv) *$\sigma(A_0) \setminus \{0\} \subset \mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$.*

Then u_ is stable in X_γ , and there exists $\delta > 0$ such that the unique solution $u(t)$ of (3.1) with initial value $u_0 \in X_\gamma$, satisfying $|u_0 - u_*|_\gamma < \delta$ and the compatibility condition $B(u_0) = 0$ on $\partial\Omega$, exists on \mathbb{R}_+ and converges exponentially fast in X_γ to some $u_\infty \in \mathcal{E}$ as $t \rightarrow \infty$.*

Proof. (a) The proof is similar to that of Theorem 2.1. It is based again on the reduction to normal form. We use the notation introduced above and denote as in Section 2 by P^s and P^c the projections onto $X_0^s = R(A_0)$ resp. $X^c = N(A_0)$. We first center (3.1) around u_* by setting $\bar{u} = u - u_*$, and obtain the following problem for \bar{u} .

$$\begin{cases} \partial_t \bar{u} + A_* \bar{u} = G(\bar{u}) & \text{in } \Omega, \\ B_* \bar{u} = H(\bar{u}) & \text{on } \partial\Omega, \\ \bar{u}(0) = \bar{u}_0 := u_0 - u_* & \text{in } \Omega. \end{cases} \quad (3.5)$$

Here G is defined as in Section 2, and

$$H(\bar{u}) = B_* \bar{u} - B(u_* + \bar{u}) = -[B(u_* + \bar{u}) - B(u_*) - B'(u_*)\bar{u}].$$

Exactly as in the proof of Theorem 2.1 we obtain a function $\phi \in C^1(B_{X^c}(0, \rho_0), X_1^s)$ with $\phi(0) = \phi'(0) = 0$ such that the equilibrium equation

$$\begin{aligned} A_* \psi(\zeta) &= G(\psi(\zeta)) \quad \text{in } \Omega, \quad \zeta \in U, \\ B_* \psi(\zeta) &= H(\psi(\zeta)) \quad \text{on } \partial\Omega, \quad \zeta \in U, \end{aligned}$$

for (3.5) can equivalently be expressed by

$$\begin{aligned} P^c A_* \phi(v) &= P^c G(v + \phi(v)), \\ P^s A_* \phi(v) &= P^s G(v + \phi(v)), \quad B_* \phi(v) = H(v + \phi(v)), \end{aligned} \quad (3.6)$$

for every $v \in B_{X^c}(0, \rho_0)$. We can now introduce the normal form of (3.1) for the variables

$$v := P^c(u - u_*) = P^c \bar{u}, \quad w := P^s(u - u_*) - \phi(P^c(u - u_*)) = P^s \bar{u} - \phi(P^c \bar{u}),$$

which reads as

$$\begin{cases} \partial_t v = T(v, w) & \text{in } \Omega, \\ \partial_t w + P^s A_* P^s w = R(v, w) & \text{in } \Omega, \\ B_* w = S(v, w) & \text{on } \partial\Omega, \\ v(0) = v_0, \quad w(0) = w_0 & \text{in } \Omega. \end{cases} \quad (3.7)$$

Using the equilibrium equations in (3.6) we can derive, similarly as in Section 2, the following expressions for T , R and S :

$$\begin{aligned} T(v, w) &= P^c(G(v + \phi(v) + w) - G(v + \phi(v))) - P^c A_* w, \\ R(v, w) &= P^s(G(v + \phi(v) + w) - G(v + \phi(v))) - \phi'(v)T(v, w), \\ S(v, w) &= H(v + \phi(v) + w) - B_* \phi(v) \\ &= H(v + \phi(v) + w) - H(v + \phi(v)). \end{aligned}$$

Clearly,

$$R(v, 0) = T(v, 0) = S(v, 0) = 0, \quad v \in B_{X^c}(0, \rho_0).$$

Therefore we are in the same situation as in Section 2, except that here the infinite dimensional part, i.e. the equation for w , has a nonlinear boundary condition in case $S(v, w) \not\equiv 0$.

(b) Let $0 < a \leq \infty$ and define the following function spaces on $(0, a) \times \Omega$:

$$\mathbb{E}_1(a) = H_p^1((0, a); X_0) \cap L_p((0, a); X_1), \quad \mathbb{E}_0(a) = L_p((0, a); X_0).$$

We also need spaces for the boundary values. For this purpose, we set with $\kappa_j = 1 - m_j/2m - 1/2mp$

$$Y_0 = L_p(\partial\Omega; \mathbb{C}^N), \quad Y_j = W_p^{2m\kappa_j}(\partial\Omega; \mathbb{C}^N),$$

and

$$\mathbb{F}(a) = \prod_{j=1}^m \mathbb{F}_j(a), \quad \mathbb{F}_j(a) = W_p^{\kappa_j}((0, a); Y_0) \cap L_p((0, a); Y_j), \quad j = 1, \dots, m.$$

Note that by trace theory we have

$$\begin{aligned} \mathbb{F}_j(a) &\hookrightarrow BC([0, a]; W_p^{2m\kappa_j - 2m/p}(\partial\Omega; \mathbb{C}^N)), \\ W_p^{2m\kappa_j - 2m/p}(\partial\Omega; \mathbb{C}^N) &\hookrightarrow BC(\partial\Omega; \mathbb{C}^N) \end{aligned}$$

by the condition $p > 2m + n$ and since $m_j < 2m$. The spaces $\mathbb{F}_j(a)$ are the trace spaces on the lateral boundary $(0, a) \times \partial\Omega$ of $(0, a) \times \Omega$ for the derivatives $D^\beta u$ of

order $|\beta| = m_j$ for $u \in \mathbb{E}_1(a)$.

The basic solvability theorem for the fully inhomogeneous linear problem

$$\begin{cases} \partial_t u + A_* u = f(t) & \text{in } \Omega, \quad t > 0, \\ B_* u = g(t) & \text{on } \partial\Omega, \quad t > 0, \\ u(0) = u_0 & \text{in } \Omega \end{cases} \quad (3.8)$$

in the L_p -setting reads as follows, see [12].

Proposition 3.2. *Let $a < \infty$. The linear problem (3.8) admits a unique solution $u \in \mathbb{E}_1(a)$ if and only if $f \in \mathbb{E}_0(a)$, $g \in \mathbb{F}(a)$, $u_0 \in X_\gamma$, and the compatibility condition $B_* u_0 = g(0)$ holds. There is a constant $C = C(a) > 0$ such that the estimate*

$$\|u\|_{\mathbb{E}_1(a)} \leq C(|u_0|_\gamma + \|f\|_{\mathbb{E}_0(a)} + \|g\|_{\mathbb{F}(a)})$$

holds for the solution u of (3.8).

We shall also need a variant of Proposition 3.2 for the problem

$$\begin{cases} \partial_t w + P^s A_* P^s w = f(t) & \text{in } \Omega, \quad t > 0, \\ B_* w = g(t) & \text{on } \partial\Omega, \quad t > 0, \\ w(0) = w_0 & \text{in } \Omega \end{cases} \quad (3.9)$$

on the half-line, where we assume $w_0 \in X_\gamma^s$ and $f \in L_p(\mathbb{R}_+; X_0^s)$. For this purpose we proceed as follows. Suppose first that u solves (3.8) with $u_0 = w_0$. Since $A_* P^c u = B_* P^c u = 0$ we then conclude that $w = P^s u$ solves problem (3.9). Let u_1 denote the solution of (3.8) with A_* replaced by $A_* + 1$. The spectrum of $A_0 + 1$ is contained in \mathbb{C}_+ , hence we may apply Proposition 3.1 of [19] to obtain a uniform estimate for u_1 in $\mathbb{E}_1(\infty)$. Then $u_2 = u - u_1$ solves the problem

$$\partial_t u_2 + A_* u_2 = u_1, \quad B_* u_2 = 0, \quad u_2(0) = 0.$$

As $\sigma(A_s) \subset \mathbb{C}_+$, A_s has maximal L_p -regularity on the half-line, hence we obtain also a uniform estimate for $P^s u_2$ in $\mathbb{E}_1(\infty)$. These arguments yield the following result.

Proposition 3.3. *Let $a \leq \infty$. The linear problem (3.9) admits a unique solution $w \in \mathbb{E}_1(a) \cap L_p((0, a); X_0^s)$ if and only if $f \in L_p((0, a); X_0^s)$, $g \in \mathbb{F}(a)$, $w_0 \in X_\gamma^s$, and the compatibility condition $B_* w_0 = g(0)$ holds. There is a constant $C_0 > 0$, independent of a , such that the estimate*

$$\|w\|_{\mathbb{E}_1(a)} \leq C_0(|w_0|_\gamma + \|f\|_{\mathbb{E}_0(a)} + \|g\|_{\mathbb{F}(a)})$$

holds for the solution w of (3.9), for all functions $f \in L_p((0, a); X_0^s)$, $g \in \mathbb{F}(a)$ and all initial values $w_0 \in X_\gamma^s$.

Proposition 3.3 remains valid when we replace $w(t)$ by $e^{\sigma t} w(t)$, $f(t)$ by $e^{\sigma t} f(t)$ and $g(t)$ by $e^{\sigma t} g(t)$ where $0 < \sigma \leq \omega$, $\omega < \inf\{\operatorname{Re} \lambda : \lambda \in \sigma(A_s)\}$.

(c) Next we consider the nonlinearities R , T , and S . Since by assumption the functions a_α and f are in C^1 and $p > n + 2m$, it follows easily via the embedding

$X_\gamma \hookrightarrow BC(\bar{\Omega})$ that A and F are as in Section 2. Hence we obtain as there the estimates

$$|T(v, w)|_0 \leq C_2(\eta + r)|w|_1 + C_3|w|_1, \quad v \in B_{X^c}(0, \rho), \quad w \in B_{X_\gamma^s}(0, \rho) \cap X_1,$$

where $r = 3\rho$ and $C_3 := \|P^c A_* P^c\|_{\mathcal{B}(X_1, X^c)}$. Since $\phi'(0) = 0$ we can assume that ρ_0 was chosen so small that $|\phi'(w)|_{\mathcal{B}(X^c, X_1^s)} \leq \eta$ for all $w \in B_{X^c}(0, \rho_0)$. With this we obtain

$$|R(v, w)|_0 \leq C_4(\eta + r)|w|_1, \quad v \in \bar{B}_{X^c}(0, \rho), \quad w \in \bar{B}_{X_\gamma^s}(0, \rho) \cap X_1.$$

Observe that in contrast to the previous section the constant C_3 is no longer small since P^l , $l \in \{c, s\}$, and A_* do not commute. However, this does not alter our conclusions. It is more involved to derive the estimates on S needed for Proposition 3.3. Fortunately, we can refer to [19, Proposition 3.3]. This result implies

$$\|e^{\omega t}(H(\bar{u}_1) - H(\bar{u}_2))\|_{\mathbb{F}(a)} \leq \eta \|e^{\omega t}(\bar{u}_1 - \bar{u}_2)\|_{\mathbb{E}_1(a)},$$

for all $e^{\omega t}\bar{u}_1, e^{\omega t}\bar{u}_2 \in \mathbb{E}_1(a)$ such that $|\bar{u}_1(t)|_\gamma, |\bar{u}_2(t)|_\gamma \leq r$, $t \in [0, a]$. Therefore, by possibly decreasing $r > 0$, with $\bar{u}_1 = v + \phi(v) + w$ and $\bar{u}_2 = v + \phi(v)$ this yields

$$\|e^{\omega t}S(v, w)\|_{\mathbb{F}(a)} \leq \eta \|e^{\omega t}w\|_{\mathbb{E}_1(a)},$$

for all $e^{\omega t}v, e^{\omega t}w \in \mathbb{E}_1(a)$ such that $v([0, a]) \subset B_{X^c}(0, \rho)$ and $w([0, a]) \subset B_{X_\gamma^s}(0, \rho)$. These are the estimates we need for applying Proposition 3.3.

(d) We may now follow part (c)-(f) of the proof of Theorem 2.1 to complete the proof of Theorem 3.1, the needed local well-posedness result being Proposition 4.1 in [19]. \square

4. CONVERGENCE OF SOLUTIONS FOR THE MULLINS-SEKERKA PROBLEM

We consider the two-phase quasi-stationary Stefan problem with surface tension, which has also been termed Mullins-Sekerka model (or Hele-Shaw model with surface tension) and is a model for phase transitions in liquid-solid systems. Let Ω be a bounded domain in \mathbb{R}^n , $n \geq 2$, with smooth boundary $\partial\Omega$. Let $\Gamma_0 \subset \Omega$ be a compact connected hypersurface in Ω which is the boundary of an open set $\Omega_0 \subset \Omega$, and let $\Gamma(t)$ be its position at time $t \geq 0$. Denote by $V(t, \cdot)$ and $\kappa(t, \cdot)$ the normal velocity and the mean curvature of $\Gamma(t)$, and let $\Omega_1(t)$ (liquid phase) and $\Omega_2(t)$ (solid phase) be the two regions in Ω separated by $\Gamma(t)$, with $\Omega_1(t)$ being the interior region. Let further $\nu(t, \cdot)$ be the outer unit normal field on $\Gamma(t)$ with respect to $\Omega_1(t)$. We shall use the convention that V is positive if $\Omega_1(t)$ is expanding, and that the mean curvature is positive for uniformly convex hypersurfaces. The *two-phase Mullins-Sekerka problem* consists in finding a family $\Gamma(t)$, $t \geq 0$, of hypersurfaces satisfying

$$V = [\partial_\nu u_\kappa], \quad t > 0, \quad \Gamma(0) = \Gamma_0, \quad (4.1)$$

where $u_\kappa = u_\kappa(t, \cdot)$ is, for each $t \geq 0$, the solution of the elliptic boundary value problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega_1(t) \cup \Omega_2(t), \\ u = \kappa & \text{on } \Gamma(t), \\ \partial_\nu u = 0 & \text{on } \partial\Omega. \end{cases} \quad (4.2)$$

Here $[\partial_\nu u_\kappa] := \partial_\nu u_\kappa^2 - \partial_\nu u_\kappa^1$ stands for the jump of the normal derivative of u_κ across the interface $\Gamma(t)$, and $\partial_\nu u$ denotes the normal derivative of u on $\partial\Omega$.

Assuming connected phases and that the interface does not touch the fixed boundary $\partial\Omega$, the set of equilibrium states of (4.1), (4.2) consists precisely of all spheres $S_R(x_0) \subset \Omega$, where R denotes the radius and x_0 the center. Thus there is an $(n+1)$ -parameter family of equilibria, the parameters being the n coordinates of the center x_0 and the radius R .

Let now $\Sigma \subset \Omega$ be some fixed sphere without boundary contact. We are interested in the asymptotic properties of solutions of the Mullins-Sekerka problem that start in a neighbourhood of Σ , that is Γ_0 is close to Σ . Following [15] we first use Hanzawa's method to transform the original problem to a system of equations on a fixed domain. Here the basic idea is to represent the moving interface $\Gamma(t)$ as the graph of a function in normal direction of a fixed reference surface, which will be Σ in our case. Denoting the parameterizing function by $\rho(t, \cdot)$ this leads to a problem on Σ of the form

$$\dot{\rho} + B(\rho)S(\rho) = 0, \quad t > 0, \quad \rho(0) = \rho_0, \quad (4.3)$$

where $S(\rho)$ is the solution of the transformed elliptic boundary value problem

$$\begin{cases} A(\rho)v = 0 & \text{in } \Omega_1 \cup \Omega_2, \\ v = K(\rho) & \text{on } \Sigma, \\ \partial_\nu v = 0 & \text{on } \partial\Omega. \end{cases} \quad (4.4)$$

Here Ω_1 and Ω_2 are the two regions in Ω separated by Σ , with Ω_1 being enclosed by Σ . By construction, the solution $\Gamma(\cdot) \equiv \Sigma$ of the original problem (4.1) corresponds to the solution $\rho \equiv 0$ of (4.3). The operator $F(\cdot) := B(\cdot)S(\cdot)$ in (4.3) is a nonlocal pseudo-differential operator of third order and renders (4.3) a quasilinear parabolic problem, see [15] for its precise definition and more details.

We want to study (4.3) in an L_p setting. Let $p > n+2$ and define

$$X_0 = W_p^{1-1/p}(\Sigma), \quad X_1 = W_p^{4-1/p}(\Sigma).$$

Given $J = (0, a)$, $a > 0$, we view (4.3) as an evolution equation in the space $\mathbb{E}_0(J) = L_p(J; X_0)$, that is we are interested in solutions of (4.3) in the class $\mathbb{E}_1(J) = H_p^1(J; X_0) \cap L_p(J; X_1)$. For the corresponding trace space we have

$$X_\gamma := (X_0, X_1)_{1-1/p, p} = W_p^{4-4/p}(\Sigma).$$

Note that, by Sobolev embedding, we have $X_1 \hookrightarrow X_\gamma \hookrightarrow C^2(\Sigma)$.

Let

$$\mathfrak{A} := \{\rho \in C^2(\Sigma) : |\rho|_{C(\Sigma)} < \eta\}$$

with $\eta > 0$ sufficiently small denote the set of admissible parameterizations. Setting $U := X_1 \cap \mathfrak{A}$, one has $F \in C^\infty(U; X_0)$ and the linearization $L := F'(0)$ is given by

$$L\rho = -[\partial_\nu T A_\Sigma \rho],$$

where Tg denotes the solution of the elliptic problem

$$\begin{cases} \Delta v = 0 & \text{in } \Omega_1 \cup \Omega_2, \\ v = g & \text{on } \Sigma, \\ \partial_\nu v = 0 & \text{on } \partial\Omega, \end{cases} \quad (4.5)$$

and

$$A_\Sigma = -\frac{1}{n-1} \left(\frac{n-1}{R^2} + \Delta_\Sigma \right),$$

with R being the radius of Σ and Δ_Σ the Laplace-Beltrami operator on Σ . The operator A_Σ is the linearization $K'(0)$ of the transformed mean curvature operator $K(\rho)$ at $\rho = 0$. Concerning the linearization $L = F'(0)$ we refer to [15, 26].

One can show (cf. [26, Theorem 2.1]) that the spectrum of L consists of countably many real nonnegative eigenvalues of finite algebraic multiplicity, and that 0 is a semi-simple eigenvalue of L with multiplicity $n+1$, see also [15, Proposition 5.4 and Lemma 6.1]. Moreover, the kernel of L is given by $N(L) = \text{span}\{Y_0, Y_1, \dots, Y_n\}$, where $Y_0 \equiv 1$, and where Y_j , $1 \leq j \leq n$, are the spherical harmonics of degree 1. We may assume that $Y_j = R^{-1}p_j|_\Sigma$, $1 \leq j \leq n$, with p_j being the harmonic polynomial of degree 1 given by $p_j(x) = x_j$ for $x \in \mathbb{R}^n$; by $p_j|_\Sigma$ we mean the restriction of p_j to Σ .

Let us assume that Σ is centered at the origin of \mathbb{R}^n . Suppose $\mathcal{S} \subset \Omega$ is a sphere that is sufficiently close to Σ . Denote by (z_1, \dots, z_n) the coordinates of its center and let z_0 be such that $R + z_0$ corresponds to its radius. Then, by [15, Section 6], the sphere \mathcal{S} can be parameterized over Σ by the distance function

$$\rho(z) = \sum_{j=1}^n z_j Y_j - R + \sqrt{\left(\sum_{j=1}^n z_j Y_j\right)^2 + (R + z_0)^2 - \sum_{j=1}^n z_j^2}.$$

Denoting by O a sufficiently small neighbourhood of 0 in \mathbb{R}^{n+1} , the mapping $[z \mapsto \rho(z)] : O \rightarrow W_p^{4-1/p}(\Sigma)$ is smooth and the derivative at 0 is given by

$$\rho'(0)h = \sum_{j=0}^n h_j Y_j, \quad h \in \mathbb{R}^{n+1}.$$

So we see that near Σ the set \mathcal{E} of equilibria of (4.3) is a smooth manifold in X_1 of dimension $n+1$, and that the tangent space $T_\Sigma(\mathcal{E})$ coincides with $N(L)$.

In order to be able to apply Theorem 2.1 from Section 2 it remains to verify that the operator L has the property of maximal L_p -regularity. This means we have to show that for any $J = (0, a)$, $a > 0$, and any $g \in \mathbb{E}_0(J)$ the problem

$$\dot{\rho} + L\rho = g, \quad t \in J, \quad \rho(0) = 0, \quad (4.6)$$

has a unique solution in the space $\mathbb{E}_1(J)$. By means of the standard localization method, perturbation arguments, and by solving certain elliptic auxiliary problems, (4.6) can be reduced to the following two-phase problem on $\mathbb{R}^n \times \dot{\mathbb{R}}$ with $\dot{\mathbb{R}} = \mathbb{R} \setminus \{0\}$:

$$-\Delta_x w - \partial_y^2 w = 0, \quad t \in J, x \in \mathbb{R}^n, y \in \dot{\mathbb{R}}, \quad (4.7)$$

$$w|_{y=0} + \Delta_x \sigma = 0, \quad t \in J, x \in \mathbb{R}^n, \quad (4.8)$$

$$\partial_t \sigma - [\partial_y w] = h, \quad t \in J, x \in \mathbb{R}^n, \quad (4.9)$$

$$\sigma(0) = 0, \quad x \in \mathbb{R}^n.$$

Here $[\partial_y w] = \partial_y w|_{y=0^+} - \partial_y w|_{y=0^-}$, and $h \in L_p(J; W_p^{1-1/p}(\mathbb{R}^n))$ is a given function. We take the Fourier transform w.r.t. x and denote the transformed functions by \tilde{w} and $\tilde{\sigma}$. Then (4.7) and (4.8) imply that $\tilde{w} = e^{-|\xi||y|}|\xi|^2 \tilde{\sigma}$. Inserting this into (4.9) leads to the subsequent problem for $\tilde{\sigma}$ on \mathbb{R}^n :

$$\partial_t \tilde{\sigma} + 2|\xi|^3 \tilde{\sigma} = \tilde{h}, \quad t \in J, \quad \tilde{\sigma}(0) = 0.$$

Set $Y_0 = W_p^{1-1/p}(\mathbb{R}^n)$ and $Y_1 = W_p^{4-1/p}(\mathbb{R}^n)$ and let G be defined by $G = d/dt$ with domain $D(G) = {}_0H_p^1(J; Y_0)$, here the zero means vanishing trace at $t = 0$. Then G is sectorial, invertible and admits an \mathcal{H}^∞ -calculus in $L_p(J; Y_0)$ of angle $\pi/2$. Let further D be the operator in $L_p(J; Y_0)$ with symbol $2|\xi|^3$ and domain $D(D) = L_p(J; Y_1)$. Then D is sectorial and admits an \mathcal{H}^∞ -calculus in $L_p(J; Y_0)$ of angle 0. Thus by the Dore-Venni theorem, the equation $G\sigma + D\sigma = h$ possesses a unique solution $\sigma \in D(G) \cap D(D)$. Hence L has the property of maximal L_p -regularity.

So all assumptions of Theorem 2.1 are satisfied, hence we obtain the following result, which is the main result in [15] except for the different functional analytic setting.

Theorem 4.1. *Let $p > n + 2$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary of class C^2 . Suppose Σ is an arbitrary sphere in Ω of radius R without boundary contact. Then $\rho \equiv 0$ is a stable equilibrium of (4.3) in $X_\gamma = W_p^{4-4/p}(\Sigma)$, and there exists $\delta > 0$ such that if $|\rho_0|_\gamma < \delta$, then the corresponding solution of (4.3) exists globally and converges at an exponential rate in X_γ to some equilibrium ρ_∞ as $t \rightarrow \infty$. In this sense, the sphere Σ is a stable equilibrium of the Mullins-Sekerka problem, and any solution $\Gamma(\cdot)$ of (4.1) that starts sufficiently close to Σ exists globally and converges to some sphere at an exponential rate as $t \rightarrow \infty$.*

This approach can also be used to show the stability of spheres for the two-phase quasi-stationary Stokes flow in a bounded domain, see [17, 16] for alternate approaches in the one-phase case. Moreover, it can be applied to models in tumor growth, see [9] for a discussion of existing work.

5. STABILITY OF TRAVELLING WAVE SOLUTIONS TO A QUASILINEAR PARABOLIC EQUATION

The situation of the generalized principle of linearized stability may occur when studying the stability of travelling wave solutions of parabolic equations, see e.g. [18, Section 5.4] for the semilinear case. In what follows we want to consider a quasilinear variant of the Huxley equation:

$$u_t - (\sigma(u_x))_x = f(u), \quad t > 0, x \in \mathbb{R}. \quad (5.1)$$

Here $f(r) = r(1-r)(r-a)$, $r \in \mathbb{R}$, where $a \in (0, 1/2)$ is a constant, and σ is a C^2 smooth function on \mathbb{R} satisfying

$$0 < c_1 \leq \sigma'(r) \leq c_2, \quad r \in \mathbb{R}. \quad (5.2)$$

A travelling wave $u(t, x) = w(x + Vt)$ with speed V satisfies

$$(\sigma(w'(s)))' - Vw'(s) + f(w(s)) = 0, \quad s \in \mathbb{R}. \quad (5.3)$$

Similarly to the special case $\sigma(r) = r$ (cf. [18]), one can show, by means of a phase plane analysis, that for some $V > 0$ (5.3) admits a solution w with $w(s) \rightarrow 0$ as $s \rightarrow -\infty$ and $w(s) \rightarrow 1$ as $s \rightarrow \infty$. For this purpose we introduce the variable $z := w'$. Then (5.3) is equivalent to the system

$$\begin{cases} \dot{w} = z, \\ \dot{z} = \frac{1}{\sigma'(z)} (Vz - f(w)). \end{cases}$$

Denoting its right-hand side by $H(w, z)$, we find that

$$H'(i, 0) = \begin{bmatrix} 0 & 1 \\ -\frac{f'(i)}{\sigma'(0)} & \frac{V}{\sigma'(0)} \end{bmatrix}, \quad i \in \{0, 1\}.$$

We have $f'(0) = -a$ and $f'(1) = a - 1$, thus the equilibria $(0, 0)$ and $(1, 0)$ are both saddle points. The eigenvalues of $H'(0, 0)$ are given by

$$\lambda_{1,2} = \frac{1}{2\sigma'(0)} \left(V \pm \sqrt{V^2 + 4a\sigma'(0)} \right),$$

$(1/\lambda_1, 1)$ is an eigenvector to $\lambda_1 > 0$, thus the unstable manifold points into the first and third quadrant, with steeper slopes for higher values of $V \geq 0$.

Define the functions

$$F(y) = \int_0^y f(r) dr, \quad G(y) = \int_0^y \sigma'(r) r dr, \quad y \in \mathbb{R}.$$

Then (5.3) implies that

$$\frac{d}{ds} \left(G(z(s)) + F(w(s)) \right) = Vz(s)^2, \quad s \in \mathbb{R}. \quad (5.4)$$

In particular $G(z) + F(w)$ is a first integral if $V = 0$. By (5.2) we further have $C_1 y^2 \leq 2G(y) \leq C_2 y^2$, $y \in \mathbb{R}$.

We now consider the trajectory γ that (near the origin) lies on the unstable manifold to $(0, 0)$ in the first quadrant. For $V = 0$, γ cannot reach the line $w = 1$, since $F(w) \leq 0$ on γ , and $F(1) = \frac{1}{6}(\frac{1}{2} - a) > 0$. In case $V > 0$, (5.4) shows that γ , as long as it remains in the first quadrant, moves through increasing values c of the level curves $G(z) + F(w) = c$. For V sufficiently large, γ will reach the level curve to $c = F(1)$ at some point with $z > 0$. For continuity reasons, there exists then $V > 0$ for which γ becomes a heteroclinic orbit, connecting $(0, 0)$ and $(1, 0)$; observe that γ is the only such orbit. Hence there is a smooth solution w to (5.3) satisfying $w'(s) > 0$ for all $s \in \mathbb{R}$ and

$$(w(s), w'(s)) \rightarrow (0, 0) \text{ as } s \rightarrow -\infty, \quad (w(s), w'(s)) \rightarrow (1, 0) \text{ as } s \rightarrow \infty, \quad (5.5)$$

both exponentially fast. Clearly each translate $w(\cdot + \alpha)$ with $\alpha \in \mathbb{R}$ enjoys the same properties.

In order to investigate the stability of the travelling wave we change to moving coordinates with $y := x + Vt$ and $\tilde{u}(t, y) := u(t, x)$, thereby transforming (5.1) into

$$\tilde{u}_t - (\sigma(\tilde{u}_y))_y + V\tilde{u}_y = f(\tilde{u}), \quad t > 0, \quad y \in \mathbb{R}. \quad (5.6)$$

Evidently, $\tilde{u} = w$ and all translates of it are equilibria of (5.6). Letting $v := \tilde{u} - w$ be the deviation from w , the equation for v reads as

$$v_t - (\sigma(v_y + w_y))_y + V(v_y + w_y) = f(v + w), \quad t > 0, \quad y \in \mathbb{R}. \quad (5.7)$$

For this equation, the set

$$\mathcal{E} = \{w(\cdot + \alpha) - w(\cdot) : \alpha \in \mathbb{R}\} \quad (5.8)$$

forms a one-dimensional smooth manifold of equilibria. Observe that \mathcal{E} contains all equilibria $w(y)$ of (5.7) satisfying $w(y) \rightarrow 0$ as $|y| \rightarrow \infty$. By definition, the travelling wave under study is stable, if $v = 0$ is stable for (5.7). Of course, this has to be understood in the sense of a suitable functional analytic setting.

Let us choose again the L_p setting to study (5.7). Let $1 < p < \infty$, $X_0 = L_p(\mathbb{R})$, and $X_1 = H_p^2(\mathbb{R})$. Define $M : X_1 \rightarrow X_0$ by

$$M(v) = -(\sigma(v_y + w_y))_y + V(v_y + w_y) - f(v + w), \quad v \in X_1,$$

This definition makes sense since by (5.2)–(5.3) and (5.5) w_y , w_{yy} , and $f(w)$ belong to $L_p(\mathbb{R})$. The fact that $f(v + w)$ belongs to the space $L_p(\mathbb{R})$ can be justified by observing that

$$f(v(y) + w(y)) = f(w(y)) + \int_0^1 f'(w(y) + \tau v(y)) d\tau v(y), \quad y \in \mathbb{R}.$$

Since v and w belong to $C_0(\mathbb{R})$ one readily verifies that $f'(v + w)$ is continuous and bounded, and this yields the statement for $f(v + w)$. The linearization $A_0 := M'(0)$ of M at $v = 0$ is given by

$$A_0 v = -(\sigma'(w_y)v_y)_y + Vv_y - f'(w)v, \quad v \in D(A_0) := X_1.$$

By (5.2), A_0 is a uniformly elliptic operator with smooth coefficients whose leading coefficient satisfies $-\sigma'(w'(y)) \rightarrow -\sigma'(0)$ as $|y| \rightarrow \infty$. Thus, by [11, Theorem 5.7], A_0 enjoys the property of maximal L_p -regularity.

Next observe that $\mathcal{E} \subset H_p^3(\mathbb{R}) \hookrightarrow X_1$ and that the tangent space for the manifold \mathcal{E} at $v = 0$ coincides with $\text{span}\{w'\}$, the span of $w' \in X_1$. On the other hand, by differentiating (5.3) we see that $A_0 w' = 0$. So to show normal stability of $v = 0$, it remains to prove that 0 is a simple eigenvalue of A_0 and that the remainder of the spectrum of A_0 lies in $\{z \in \mathbb{C} : \text{Re } z > 0\}$. We proceed similarly as in [18, p. 131], generalizing the proof given there to the quasilinear case. Suppose λ with $\text{Re } \lambda \leq 0$ is an eigenvalue of A_0 with eigenfunction $v \in X_1$, that is

$$v'' + 2b(y)v' + \frac{f'(w(y))}{\sigma'(w'(y))}v + \frac{\lambda v}{\sigma'(w'(y))} = 0, \quad y \in \mathbb{R},$$

where

$$b(y) = \frac{\sigma''(w'(y))w''(y) - V}{2\sigma'(w'(y))}, \quad y \in \mathbb{R}.$$

By studying the characteristic equation for the limits $y \rightarrow \pm\infty$, one sees that there exist constants $\delta, C > 0$ such that

$$|v(y)| \leq Ce^{(\delta+V/\sigma'(0))y}, \quad y \leq 0, \quad |v(y)| \leq Ce^{-\delta y}, \quad y \geq 0.$$

Letting $\varphi(y) = v(y) \exp(\int_0^y b(r) dr)$, we have at least $\varphi(y) = O(e^{-V|y|/(2\sigma'(0))})$ for $|y| \rightarrow \infty$, and

$$\varphi'' + \left(\frac{\lambda}{\sigma'(w'(y))} + \frac{f'(w(y))}{\sigma'(w'(y))} - b'(y) - b(y)^2 \right) \varphi = 0, \quad y \in \mathbb{R}. \quad (5.9)$$

The function $\psi(y) := w'(y) \exp(\int_0^y b(r) dr)$ is strictly positive, and since $A_0 w' = 0$, it satisfies

$$\psi'' + \left(\frac{f'(w(y))}{\sigma'(w'(y))} - b'(y) - b(y)^2 \right) \psi = 0, \quad y \in \mathbb{R}. \quad (5.10)$$

Combining (5.9), (5.10) yields

$$\varphi'' + \frac{\lambda}{\sigma'(w'(y))} \varphi - \frac{\psi''}{\psi} \varphi = 0.$$

Multiplying this equation by $\bar{\varphi}$, integrating over \mathbb{R} , and integrating by parts gives

$$\begin{aligned} \lambda \int_{\mathbb{R}} \frac{|\varphi(y)|^2}{\sigma'(w'(y))} dy &= \int_{\mathbb{R}} -\frac{\bar{\varphi}(y)}{\psi(y)} (\varphi''(y)\psi(y) - \varphi(y)\psi''(y)) dy \\ &= \int_{\mathbb{R}} \psi(y)^2 \left\{ [(\text{Re } \varphi(y)/\psi(y))']^2 + [(\text{Im } \varphi(y)/\psi(y))']^2 \right\} dy, \end{aligned}$$

hence $\lambda = 0$. In view of (5.9) we may then suppose that φ is real and nonnegative, and so the last formula above implies that $\varphi/\psi = v/w'$ is constant on \mathbb{R} . Hence $N(A_0) = \text{span}\{w'\}$.

We now show that the essential spectrum $\sigma_e(A_0)$ of A_0 , that is the set of all spectral points except isolated eigenvalues of finite multiplicity, is contained in $[\text{Re } z > 0]$. This can be seen as follows.

By the asymptotics of w , see (5.5), we know that $f'(w(y)) \rightarrow f'(0) = -a$ as $y \rightarrow -\infty$, and that $f'(w(y)) \rightarrow f'(1) = -(1-a)$ as $y \rightarrow \infty$ at an exponential rate. Due to $a \in (0, 1/2)$, a is the smaller of the two numbers $\{a, (1-a)\}$. Fix $\varepsilon > 0$

so that $a - \varepsilon > 0$. We can then find a number $R > 0$ so that $-f'(w(y)) \geq a - \varepsilon$ whenever $|y| \geq R$. Let c be a bounded continuous function on \mathbb{R} that agrees with $-f'(w)$ on $[|y| \geq R]$ and satisfies $c(y) \geq a - \varepsilon$ for all $y \in \mathbb{R}$. Let $B : X_1 \rightarrow X_0$ be the operator defined by

$$Bv := -(\sigma'(w_y)v_y)_y + Vv_y + cv.$$

One readily shows that B is accretive on $L_p(\mathbb{R})$ and also that $\sigma(B)$, the spectrum of B , is contained in $[\operatorname{Re} z \geq (a - \varepsilon)]$. Next, note that A_0 can be written as $A_0 = B + S$, where S is a perturbation which is relatively compact with respect to B . For this we note that S can be written as $Sv = -(f'(w) + c)\chi v$, where χ is a smooth cut-off function for the interval $[-R, R]$, with support contained, say, in $\Omega = (-2R, 2R)$. Since $H_p^s(\Omega)$ is compactly embedded in $L_p(\Omega)$ for $s > 0$ we conclude that S is a compact operator from $H_p^s(\mathbb{R}) \rightarrow L_p(\mathbb{R})$. A well-known perturbation result now shows that $\sigma_e(A_0)$ must also be contained in $[\operatorname{Re} z \geq (a - \varepsilon)]$. Since this is true for every ε , we have proved that $\sigma_e(A_0) \subset [\operatorname{Re} z \geq a]$. A_0 might still have isolated eigenvalues outside of this set. As A_0 generates an analytic semigroup, they must be contained in an appropriate sector with opening angle $\theta < \pi/2$. Since we have already shown that there are no eigenvalues in $[\operatorname{Re} z < 0]$ we conclude that there is a number $\alpha > 0$ so that $\sigma(A_0) \setminus \{0\} \subset [\operatorname{Re} z \geq \alpha]$. Finally, since the operator A_1 defined by

$$A_1v := v_{yy} + \left(\frac{f'(w)}{\sigma'(w')} - b'(y) - b(y)^2 \right)v, \quad v \in D(A_1) = X_1,$$

is self-adjoint in $L_2(\mathbb{R})$, it follows in view of (5.9) with $\lambda = 0$ that the eigenvalue 0 of A_0 is semi-simple.

Summarizing we have shown that

- the set \mathcal{E} consists of all equilibria of (5.7) in X_1 and forms a smooth 1-dimensional manifold,
- $T_0(\mathcal{E}) = \operatorname{span}\{w'\} = N(A_0)$,
- the eigenvalue 0 is semi-simple,
- $\sigma(A_0) \setminus \{0\} \subset \mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$,

and we can now formulate our main result for this section.

Theorem 5.1. *Let $p \in (1, \infty)$, $f(r) = r(1 - r)(r - a)$, $r \in \mathbb{R}$, with $a \in (0, 1/2)$, and $\sigma \in C^2(\mathbb{R})$ such that (5.2) holds.*

Then (5.1) possesses a travelling wave solution $u(t, x) = w(x + Vt)$ with speed $V > 0$ and profile $w \in C^3(\mathbb{R})$ satisfying $w'(r) > 0$, $r \in \mathbb{R}$, and $(w(r), w'(r)) \rightarrow (0, 0)$ as $r \rightarrow -\infty$, as well as $(w(r), w'(r)) \rightarrow (1, 0)$ as $r \rightarrow \infty$.

This travelling wave is stable in the sense that $v \equiv 0$ is a stable equilibrium of (5.7) in $X_\gamma = W_p^{2-2/p}(\mathbb{R})$. Moreover there exists $\delta > 0$ such that if $|v_0|_\gamma < \delta$, then the solution v of (5.7) with $v(0) = v_0$ exists globally and converges at an exponential rate in X_γ to some equilibrium v_∞ , i.e. to some element of the set \mathcal{E} defined in (5.8). In this sense, any solution u of (5.1) that starts sufficiently

close to w exists globally and converges at an exponential rate as $t \rightarrow \infty$ to some translate $w(x + Vt + \alpha)$, $\alpha \in \mathbb{R}$, of the travelling wave solution.

This approach applies to many other travelling wave solutions of quasilinear parabolic systems, as soon as the condition of normal stability is satisfied.

6. CONVERGENCE FOR ABSTRACT QUASILINEAR PROBLEMS II

We return to the setting of Section 2 for the case that $\sigma(A_0)$ also contains an unstable part, i.e. we now assume that

$$\sigma(A_0) = \{0\} \cup \sigma_s \cup \sigma_u, \quad \text{with } \sigma_s \subset \mathbb{C}_+, \sigma_u \subset \mathbb{C}_-, \quad (6.1)$$

such that $\sigma_u \neq \emptyset$. In this situation we can prove the following result.

Theorem 6.1. *Let $1 < p < \infty$. Suppose $u^* \in V \cap X_1$ is an equilibrium of (2.1), and suppose that the functions (A, F) satisfy (2.2). Suppose further that $A(u^*)$ has the property of maximal L_p -regularity. Let A_0 be the linearization of (2.1) at u_* . Suppose that u_* is normally hyperbolic, i.e. assume that*

- (i) *near u_* the set of equilibria \mathcal{E} is a C^1 -manifold in X_1 of dimension $m \in \mathbb{N}$,*
- (ii) *the tangent space for \mathcal{E} at u_* is given by $N(A_0)$,*
- (iii) *0 is a semi-simple eigenvalue of A_0 , i.e. $N(A_0) \oplus R(A_0) = X_0$,*
- (iv) *$\sigma(A_0) \cap i\mathbb{R} = \{0\}$, $\sigma_u := \sigma(A_0) \cap \mathbb{C}_- \neq \emptyset$.*

Then u_ is unstable in X_γ and even in X_0 .*

For each sufficiently small $\rho > 0$ there exists $0 < \delta \leq \rho$ such that the unique solution $u(t)$ of (2.1) with initial value $u_0 \in B_{X_\gamma}(u_, \delta)$ either satisfies*

- *$\text{dist}_{X_\gamma}(u(t_0), \mathcal{E}) > \rho$ for some finite time $t_0 > 0$, or*
- *$u(t)$ exists on \mathbb{R}_+ and converges at an exponential rate to some $u_\infty \in \mathcal{E}$ in X_γ as $t \rightarrow \infty$.*

Proof. The first assertion follows from [24, Theorem 6.2], so we need to prove the second claim.

(a) Let P^l denote the spectral projections corresponding to the spectral sets σ_l , where $\sigma_c = \{0\}$ and σ_s, σ_u are as in (6.1). Let $X_j^l = P^l(X_j)$, $l \in \{c, s, u\}$, where these spaces are equipped with the norms of X_j for $j \in \{0, 1, \gamma\}$. We may assume that X_1 is equipped with the graph norm of A_0 , i.e. $|v|_1 := |v|_0 + |A_0 v|_0$ for $v \in X_1$. Since the operator $-A_0$ generates an analytic C_0 -semigroup on X_0 , σ_u is a compact spectral set for A_0 . This implies that $P^u(X_0) \subset X_1$. Consequently, X_0^u and X_1^u coincide as vector spaces. In addition, since A_u , the part of A_0 in X_0^u , is invertible, we conclude that the spaces X_j^u carry equivalent norms. We set $X^u := X_0^u = X_1^u$ and equip X^u with the norm of X_0 , that is, $X^u = (X^u, |\cdot|_0)$. As in the proof of Theorem 2.1 we obtain the decomposition

$$X_1 = X^c \oplus X_1^s \oplus X^u, \quad X_0 = X^c \oplus X_0^s \oplus X^u,$$

and this decomposition reduces A_0 into $A_0 = A_c \oplus A_s \oplus A_u$, where A_l is the part of A_0 in X_0^l for $l \in \{c, s, u\}$. It follows that $\sigma(A_l) = \sigma_l$ for $l \in \{c, s, u\}$. Moreover,

due to assumption (iii), $A_c \equiv 0$. In the sequel, as a norm in X_j we take

$$|v|_j = |P^c v| + |P^s v|_j + |P^u v| \quad \text{for } j = 0, \gamma, 1. \quad (6.2)$$

We remind that the spaces X_j^l have been given the norm of X_0^l for $l \in \{c, u\}$.

We also fix constants $\omega \in (0, \inf \operatorname{Re} \sigma(-A_u))$ and $M_5 > 0$ such that $|e^{A_u t}| \leq M_5 e^{-\omega t}$ for all $t > 0$. Wlog we may take $\omega \leq 1$.

(b) Let Φ be the mapping obtained in step (b) of the proof of Theorem 2.1, and set $\phi_l(x) := P^l \Phi(x)$ for $l \in \{s, u\}$ and for $x \in B_{X^c}(0, \rho_0)$. Then

$$\phi_l \in C^1(B_{X^c}(0, \rho_0), X_1^l), \quad \phi_l(0) = \phi'_l(0) = 0 \quad \text{for } l \in \{s, u\}. \quad (6.3)$$

These mappings parametrize the manifold \mathcal{E} of equilibria near u_* via

$$[x \mapsto (x + \phi_s(x) + \phi_u(x) + u_*)], \quad x \in B_{X^c}(0, \rho_0).$$

We may assume that ρ_0 has been chosen small enough so that

$$\|\phi'_l(x)\|_{\mathcal{B}(X^c, X_1^l)} \leq 1, \quad x \in B_{X^c}(0, \rho_0), \quad l \in \{s, u\}. \quad (6.4)$$

(c) The equilibrium equation (2.7) now corresponds to the system

$$\begin{aligned} P^c G(x + \phi_s(x) + \phi_u(x)) &= 0, \\ P^l G(x + \phi_s(x) + \phi_u(x)) &= A_l \phi_l(x), \quad x \in B_{X^c}(0, \rho_0), \quad l \in \{s, u\}. \end{aligned} \quad (6.5)$$

The canonical variables are

$$x = P^c v, \quad y = P^s v - \phi_s(x), \quad z = P^u v - \phi_u(x)$$

and the canonical form of the system is given by

$$\begin{cases} \dot{x} = T(x, y, z), & x(0) = x_0, \\ \dot{y} + A_s y = R_s(x, y, z), & y(0) = y_0, \\ \dot{z} + A_u z = R_u(x, y, z), & z(0) = z_0. \end{cases} \quad (6.6)$$

Here the functions T , R_s , and R_u are given by

$$\begin{aligned} T(x, y, z) &= P^c (G(x + y + z + \phi_s(x) + \phi_u(x)) - G(x + \phi_s(x) + \phi_u(x))), \\ R_l(x, y, z) &= P^l (G(x + y + z + \phi_s(x) + \phi_u(x)) - G(x + \phi_s(x) + \phi_u(x))) \\ &\quad - \phi'_l(x) T(x, y, z), \quad l \in \{s, u\}, \end{aligned} \quad (6.7)$$

where we have used the equilibrium equations (6.5). Clearly,

$$R_l(x, 0, 0) = T(x, 0, 0) = 0, \quad x \in B_{X^c}(0, \rho_0), \quad l \in \{s, u\},$$

showing that the equilibrium set \mathcal{E} of (2.1) near u_* has been reduced to the set $B_{X^c}(0, \rho_0) \times \{0\} \times \{0\} \subset X^c \times X^s \times X^u$.

There is a unique correspondence between the solutions of (2.1) close to u_* in X_γ and those of (6.6) close to 0. We again call (6.6) the *normal form* of (2.1) near its *normally hyperbolic* equilibrium u_* .

(d) The estimates for R_l and T are similar to those derived in Section 2, and we have

$$|T(x, y, z)|, |R_l(x, y, z)|_0 \leq \beta(|y|_1 + |z|), \quad (6.8)$$

for all $x, z \in \bar{B}_X^{\tilde{l}}(0, \rho)$, $\tilde{l} \in \{c, u\}$, and $y \in \bar{B}_{X_\gamma^s}(0, \rho) \cap X_1$, where $\rho \leq \rho_0$, $r = 5\rho$, and $\beta = C_2(\eta + r)$.

(e) Let us assume for the moment that ρ is chosen so that $4\rho \leq \rho_0$. Let $u(t) = u_* + \Phi(x(t)) + y(t) + z(t)$ be a solution of (6.6) on some maximal time interval $[0, t_*)$ which satisfies $\text{dist}_{X_\gamma}(u(t), \mathcal{E}) \leq \rho$. Set

$$t_1 := t_1(x_0, y_0, z_0) := \sup\{t \in (0, t_*) : |u(\tau) - u_*|_\gamma \leq 3\rho, \tau \in [0, t]\}$$

and suppose that $t_1 < t_*$. Assuming wlog that the embedding constant of $X_1 \hookrightarrow X_\gamma$ is less or equal to one it follows from (6.2), (6.4) and the definition of t_1 that

$$|x(t)|, |y(t)|_\gamma, |z(t)| \leq 3\rho, \quad t \in [0, t_1], \quad (6.9)$$

so that the estimate (6.8) holds for $(x(t), y(t), z(t))$, $t \in [0, t_1]$.

Since \mathcal{E} is a finite-dimensional manifold, for each $u \in B_{X_\gamma}(u_*, 3\rho)$ there is $\bar{u} \in \mathcal{E}$ such that $\text{dist}_{X_\gamma}(u, \mathcal{E}) = |u - \bar{u}|_\gamma$, and by the triangle inequality $\bar{u} \in B_{X_\gamma}(u_*, 4\rho)$. Thus we may write $u = u_* + \Phi(x) + y + z$ and $\bar{u} = u_* + \Phi(\bar{x})$, and therefore

$$\begin{aligned} \rho &\geq \text{dist}_{X_\gamma}(u, \mathcal{E}) = |u - \bar{u}|_\gamma \\ &= |x - \bar{x}| + |y + \phi_s(x) - \phi_s(\bar{x})|_\gamma + |z + \phi_u(x) - \phi_u(\bar{x})| \\ &\geq |x - \bar{x}| + |z| - |\phi_u(x) - \phi_u(\bar{x})| \geq |z|, \end{aligned}$$

since $x, \bar{x} \in B_{X^c}(0, \rho_0)$ and ϕ_s is non-expansive, see (6.4). Therefore we obtain the improved estimate $|z(t)| \leq \rho$ for all $t \in [0, t_1]$.

We begin the estimates with that for the unstable component $z(t)$. Integrating the equation for z backwards yields

$$z(t) = e^{A_u(t_1-t)} z(t_1) - \int_t^{t_1} e^{A_u(s-t)} R_u(x(s), y(s), z(s)) ds. \quad (6.10)$$

With (6.8) and $|z(t_1)| \leq \rho$ we get

$$|z(t)| \leq M_5 e^{-\omega(t_1-t)} \rho + \beta M_5 \int_t^{t_1} e^{-\omega(s-t)} (|y(s)|_1 + |z(s)|) ds$$

for $t \in [0, t_1]$. Gronwall's inequality yields

$$|z(t)| \leq M_5 e^{-\omega_1(t_1-t)} \rho + \beta M_5 \int_t^{t_1} e^{-\omega_1(s-t)} |y(s)|_1 ds$$

for $t \in [0, t_1]$, where $\omega_1 = \omega - \beta M_5 > 0$ provided β , i.e. η, r are small enough. In particular, with $M_6 = M_5/\omega_1$, this inequality implies

$$\|z\|_{L_q(J_1; X_0)} \leq M_6 \rho + \beta M_6 \|y\|_{L_q(J_1; X_1)}, \quad (6.11)$$

where we have set $J_1 = (0, t_1)$; here $q \in [1, \infty]$ is arbitrary at the moment. A similar estimate holds for the time-derivative of z , namely

$$\|\dot{z}\|_{L_q(J_1; X_0)} \leq (\|A_u\| + \beta) \|z\|_{L_q(J_1; X_0)} + \beta \|y\|_{L_q(J_1; X_1)}. \quad (6.12)$$

Note that

$$\begin{aligned} |z(t+h) - z(t)| &\leq h^{1/p'} \|\dot{z}\|_{L_p(J_1; X_0)}, \\ \int_0^{t_1-h} |z(t+h) - z(t)| dt &\leq h \|\dot{z}\|_{L_1(J_1; X_0)}. \end{aligned} \quad (6.13)$$

Next we consider the equation for x . We have

$$\begin{aligned} |x(t)| &\leq |x_0| + \int_0^t |\dot{x}(s)| ds = |x_0| + \int_0^t |T(x(s), y(s), z(s))| ds \\ &\leq |x_0| + \beta(\|y\|_{L_1(J_1; X_1)} + \|z\|_{L_1(J_1; X_0)}). \end{aligned}$$

Combining this estimate with that for z we obtain

$$\begin{aligned} \sup_{t \in J_1} |x(t)| &\leq |x_0| + \|\dot{x}\|_{L_1(J_1; X_0)}, \\ \|\dot{x}\|_{L_q(J_1; X_0)} &\leq \beta(M_6 \rho + (1 + \beta M_6) \|y\|_{L_q(J_1; X_1)}). \end{aligned}$$

This estimate is best possible and shows that in order to control $|x(t)|$ we must be able to control $\|y\|_{L_1(J_1; X_1)}$. Note that

$$\begin{aligned} |x(t+h) - x(t)| &\leq h^{1/p'} \|\dot{x}\|_{L_p(J_1; X_0)}, \\ \int_0^{t_1-h} |x(t+h) - x(t)| dt &\leq h \|\dot{x}\|_{L_1(J_1; X_0)}. \end{aligned} \quad (6.14)$$

Now we turn to the equation for y , the stable but infinite dimensional part of the problem. As in the proof of Theorem 2.1, part (e), we obtain from (6.8)

$$\|y\|_{\mathbb{E}_1(t_1)} \leq M_1 |y_0|_\gamma + \beta M_0 (\|y\|_{\mathbb{E}_1(t_1)} + \|z\|_{\mathbb{E}_0(t_1)}).$$

Employing (6.11) with $q = p$ we get

$$\|y\|_{\mathbb{E}_1(t_1)} \leq M_1 |y_0|_\gamma + \beta M_0 M_6 \rho + \beta M_0 (1 + \beta M_6) \|y\|_{\mathbb{E}_1(t_1)}.$$

Assuming $\beta M_0 (1 + \beta M_6) < 1/2$, this yields

$$\|y\|_{\mathbb{E}_1(t_1)} \leq 2M_1 |y_0|_\gamma + 2\beta M_0 M_6 \rho. \quad (6.15)$$

Repeating the estimates leading up to (2.24) with $\sigma = 0$ we now get

$$|y(t)|_\gamma \leq C_5 (|y_0|_\gamma + \beta \rho), \quad t \in [0, t_1], \quad (6.16)$$

where C_5 is a constant independent of ρ , y_0 and t_1 . In particular, we see that $|y(t)|_\gamma \leq \rho$ for all $t \in J_1$, provided $|y_0|_\gamma$ and β , i.e. η and r are sufficiently small.

For later purposes we need an estimate for $|y(t+h) - y(t)|_\gamma$. We have

$$\begin{aligned} |y(t+h) - y(t)|_\gamma &\leq C |y(t+h) - y(t)|_0^{1-\gamma} |y(t+h) - y(t)|_1^\gamma \\ &\leq C h^{(1-\gamma)/p'} \|\dot{y}\|_{L_p(J_1; X_0)}^{1-\gamma} (|y(t+h)|_1^\gamma + |y(t)|_1^\gamma) \end{aligned} \quad (6.17)$$

for all $t \in [0, t_1]$, $t+h \in [0, t_1]$ with $y(t+h), y(t) \in X_1$. We remind that $\gamma = 1 - 1/p$.

Unfortunately, this is not enough to keep $|x(t)|$ small on J_1 , for this we need to control $\|y\|_{L_1(J_1; X_1)}$, and we cannot expect maximal regularity in L_1 .

To handle $\|y\|_{L_1(J_1; X_1)}$, we are forced to use another type of maximal regularity, namely that for the vector-valued Besov spaces $B_{1\infty}^\alpha(J_1; X)$, where $\alpha \in (0, 1)$; cf. [23, Theorem 7.5]. Before stating the result we remind that

$$\begin{aligned} \|g\|_{B_{1\infty}^\alpha(J; X)} &:= \|g\|_{L_1(J; X)} + [g]_{J; \alpha, X}, \\ [g]_{J; \alpha, X} &:= \sup_{0 < h \leq \min(1, a)} h^{-\alpha} \int_0^{a-h} |g(t+h) - g(t)|_X dt \end{aligned}$$

defines a norm for $g \in B_{1\infty}^\alpha(J; X)$, where $J = (0, a)$. The maximal regularity result, which is valid for all exponentially stable analytic C_0 -semigroups, reads as follows: there is a constant M_7 depending only on A_s and on $\alpha \in (0, 1)$ such that the solution y of

$$\dot{y} + A_s y = f, \quad t \in J, \quad y(0) = y_0, \quad (6.18)$$

satisfies the estimate

$$\|y\|_{B_{1\infty}^\alpha(J; X_1^s)} \leq M_7(|y_0|_{D_{A_s}(\alpha, \infty)} + \|f\|_{B_{1\infty}^\alpha(J; X_0^s)}).$$

Note that this estimate is in particular independent of $J = (0, a)$, by exponential stability of $e^{-A_s t}$. Further we have $y_0 \in X_\gamma \cap X^s = D_{A_s}(1 - 1/p, p) \hookrightarrow D_{A_s}(\alpha, \infty)$, provided $\alpha \leq 1 - 1/p$. Another parabolic estimate valid for (6.18) that we shall make use of reads as

$$\|y\|_{B_{1\infty}^\alpha(J; X_1^s)} \leq M_8(|y_0|_{D_{A_s}(\alpha, \infty)} + \|f\|_{L_1(J_1; X_1^s)}),$$

provided $\alpha < 1$. Here the constant M_8 is also independent of $J = (0, a)$.

We set $R_1(t) = -\phi'_s(x(t))T(x(t), y(t), z(t))$ and recall that $|\phi'_s(x(t))|_{\mathcal{B}(X^c, X_1)} \leq 1$ for $t \in [0, t_1]$. Employing the L_1 -estimate for z , see (6.11), yields

$$\begin{aligned} \|R_1\|_{L_1(J_1; X_1)} &\leq \int_0^{t_1} |T(x(s), y(s), z(s))| ds \leq \beta(\|y\|_{L_1(J_1; X_1)} + \|z\|_{L_1(J_1; X_0)}) \\ &\leq \beta M_6 \rho + \beta(1 + \beta M_6) \|y\|_{L_1(J_1; X_1)}. \end{aligned}$$

Therefore, for the solution y_1 of (6.18) with $f = R_1$ we obtain

$$\|y_1\|_{B_{1\infty}^\alpha(J_1; X_1)} \leq M_8(|y_0|_\gamma + \beta M_6 \rho + \beta(1 + M_6 \beta) \|y\|_{L_1(J_1; X_1)}).$$

Next let $R_2(t) = P^s(G(\Phi(x) + y + z) - G(\Phi(x)))$. Then by estimate (2.19)

$$\begin{aligned} \|R_2\|_{L_1(J_1; X_0)} &\leq \beta(\|y\|_{L_1(J_1; X_1)} + \|z\|_{L_1(J_1; X_0)}) \\ &\leq \beta M_6 \rho + \beta(1 + M_6 \beta) \|y\|_{L_1(J_1; X_1)}, \end{aligned}$$

and with some constant C_6

$$\begin{aligned} |R_2(t) - R_2(\bar{t})|_0 &\leq C_6 \beta (|y(t) - y(\bar{t})|_1 + |z(t) - z(\bar{t})| + |x(t) - x(\bar{t})|) \\ &\quad + C_6 |y(t)|_1 (|y(t) - y(\bar{t})|_\gamma + |x(t) - x(\bar{t})| + |z(t) - z(\bar{t})|). \end{aligned}$$

Hence we obtain the following estimate

$$[R_2]_{\alpha,0} \leq C_6 \beta \{ [y]_{\alpha,1} + [z]_{\alpha,0} + [x]_{\alpha,0} \} + C_6 \sup_{0 < h \leq h_1} h^{-\alpha} \int_0^{t_1-h} |y(t)|_1 \cdot \{ |y(t+h) - y(t)|_\gamma + |x(t+h) - x(t)| + |z(t+h) - z(t)| \} dt$$

where we set $h_1 := \min(1, t_1)$ and $[\cdot]_{\alpha,j} := [\cdot]_{J_1;\alpha,X_j}$ for $j = 0, 1$. (6.11)–(6.13) yields for each $\alpha \in (0, 1)$

$$[z]_{\alpha,0} \leq \|\dot{z}\|_{L_1(J_1;X_0)} \leq C_7(\rho + \beta\|y\|_{L_1(J_1;X_1)}),$$

with some uniform constant C_7 . In the same way we may estimate $[x]_{\alpha,0}$. Next we have again by (6.11)–(6.13)

$$\begin{aligned} \sup_{0 < h \leq h_1} h^{-\alpha} \int_0^{t_1-h} |y(t)|_1 |z(t+h) - z(t)| dt &\leq h^{1/p' - \alpha} \|\dot{z}\|_{L_p(J_1;X_0)} \|y\|_{L_1(J_1;X_1)} \\ &\leq C_8(|y_0|_\gamma + \rho) \|y\|_{L_1(J_1;X_1)} \end{aligned}$$

provided $\alpha \leq 1 - 1/p$, and similarly for the corresponding integral containing the x -difference. Last but not least, for $\alpha \leq (1 - \gamma)(1 - 1/p)$ we have by (6.17)

$$\begin{aligned} \sup_{0 < h \leq h_1} h^{-\alpha} \int_0^{t_1-h} |y(t)|_1 |y(t+h) - y(t)|_\gamma dt &\leq 2Ch^{(1-\gamma)/p' - \alpha} \|\dot{y}\|_{L_p(J_1;X_0)}^{1-\gamma} \|y\|_{L_p(J_1;X_1)} \|y\|_{L_1(J_1;X_1)}^\gamma \\ &\leq C_9(|y_0|_\gamma + \beta\rho)^{2-\gamma} \|y\|_{L_1(J_1;X_1)}^\gamma \\ &\leq C_{10}((|y_0|_\gamma + \beta\rho)^2 + (|y_0|_\gamma + \beta\rho) \|y\|_{L_1(J_1;X_1)}), \end{aligned}$$

where we used Young's inequality in the last line.

Collecting now all terms and choosing $\alpha = (1 - \gamma)/p' = 1/pp'$, we find a uniform constant C_{11} such that for $|y_0|_\gamma \leq \delta$

$$\|y\|_{B_{1\infty}^\alpha(J_1;X_1^s)} \leq C_{11}(|y_0|_\gamma + \beta\rho + (\beta + \rho + \delta) \|y\|_{B_{1\infty}^\alpha(J_1;X_1^s)}),$$

hence

$$\|y\|_{L_1(J_1;X_1^s)} \leq \|y\|_{B_{1\infty}^\alpha(J_1;X_1^s)} \leq 2C_{11}(|y_0|_\gamma + \beta\rho), \quad (6.19)$$

provided $C_{11}(\beta + \rho + \delta) < 1/2$. Choosing now first β , i.e. η and r small enough, and then ρ and $\delta > 0$, we see that $|u(t_1) - u_*|_\gamma < 3\rho$, a contradiction to $t_1 < t_*$. As in (e) of the proof of Theorem 2.1 we may then conclude that $t_* = \infty$, which means that the solution exists globally and stays in the ball $\bar{B}_{X_\gamma}(u_*, 3\rho)$.

(f) To prove convergence, let $(x(t), y(t), z(t))$ be a global solution of (6.6) that satisfies

$$|x(t)|, |y(t)|_\gamma, |z(t)| \leq 3\rho, \quad \text{for all } t \geq 0,$$

see (6.9). Similarly to the proof of Theorem 2.1, part (e), we obtain from (6.8)

$$\|e^{\omega t} y\|_{\mathbb{E}_1(\infty)} \leq 2M_1 |y_0|_\gamma + 2\beta M_0 \|e^{\omega t} z\|_{\mathbb{E}_0(\infty)}, \quad (6.20)$$

where $\omega \in (0, \inf\{\operatorname{Re} \lambda : \lambda \in \sigma(A_s)\})$ is a fixed number and β is given in (2.21). Repeating the estimates leading up to (2.24) we get

$$|e^{\omega t} y(t)|_\gamma \leq M_2 |y_0|_\gamma + 2\beta c_0 M_0 \|e^{\omega t} z\|_{\mathbb{E}_0(\infty)}, \quad t \geq 0. \quad (6.21)$$

From equation (6.10) we infer that

$$z(t) = - \int_t^\infty e^{-A_u(t-s)} R_u(x(s), y(s), z(s)) ds, \quad t \geq 0, \quad (6.22)$$

since $|z(t_1)| \leq \rho$ for each $t_1 > 0$ and $e^{A_u(t_1-t)}$ is exponentially decaying for $t_1 \rightarrow \infty$. Using (6.22) and the estimate for R_u from (6.8) and proceeding as in the proof of Young's inequality for convolution integrals one shows that

$$\|e^{\omega t} z\|_{\mathbb{E}_0(\infty)} \leq C_{12} \beta (\|e^{\omega t} y\|_{\mathbb{E}_1(\infty)} + \|e^{\omega t} z\|_{\mathbb{E}_0(\infty)}). \quad (6.23)$$

Making β sufficiently small (by decreasing η and, accordingly, r) it follows from (6.20) and (6.23) that

$$\|e^{\omega t} y\|_{\mathbb{E}_1(\infty)} + \|e^{\omega t} z\|_{\mathbb{E}_0(\infty)} \leq C_{13} |y_0|_\gamma.$$

This estimate in turn, together with (6.21), implies $|y(t)|_\gamma \rightarrow 0$ and $|z(t)| \rightarrow 0$ exponentially fast as $t \rightarrow \infty$. As in the proof of Theorem 2.1 part (f) we get

$$x(t) \rightarrow x_\infty := x_0 + \int_0^\infty T(x(s), y(s), z(s)) ds.$$

This yields existence of the limit

$$u_\infty = u_* + v_\infty := u_* + \lim_{t \rightarrow \infty} v(t) = u_* + x_\infty + \phi_s(x_\infty) + \phi_u(x_\infty) \in \mathcal{E}.$$

Similar arguments as in Section 2 yield exponential convergence of $u(t)$ to u_∞ . \square

A result similar to Theorem 6.1 is also valid in the setting of Section 3. We leave the details to the interested reader.

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